A Theorem of Ramsey-Ramsey's Number

A simple instance

- *"Of 6 (or more) people, either there are 3 each pair of whom are acquainted or there are 3 each pair of whom are unacquainted"*
- Can we explain this without brute force?

- Using notations from graph theory, the above statement can be formally stated as:
  - $K_6 \rightarrow K_3, K_3$ (where K_n represents a complete graph of n points).

Graphical Abstraction

- K_6: set of 6 people and all 15 pairs of these people
- We can see K_6 by choosing 6 points (no 3 of which are collinear) and then drawing the edge connecting each pair of points.
- It is called a complete graph of order 6.
- For visualising K_3, we search for a triangle in the graph.

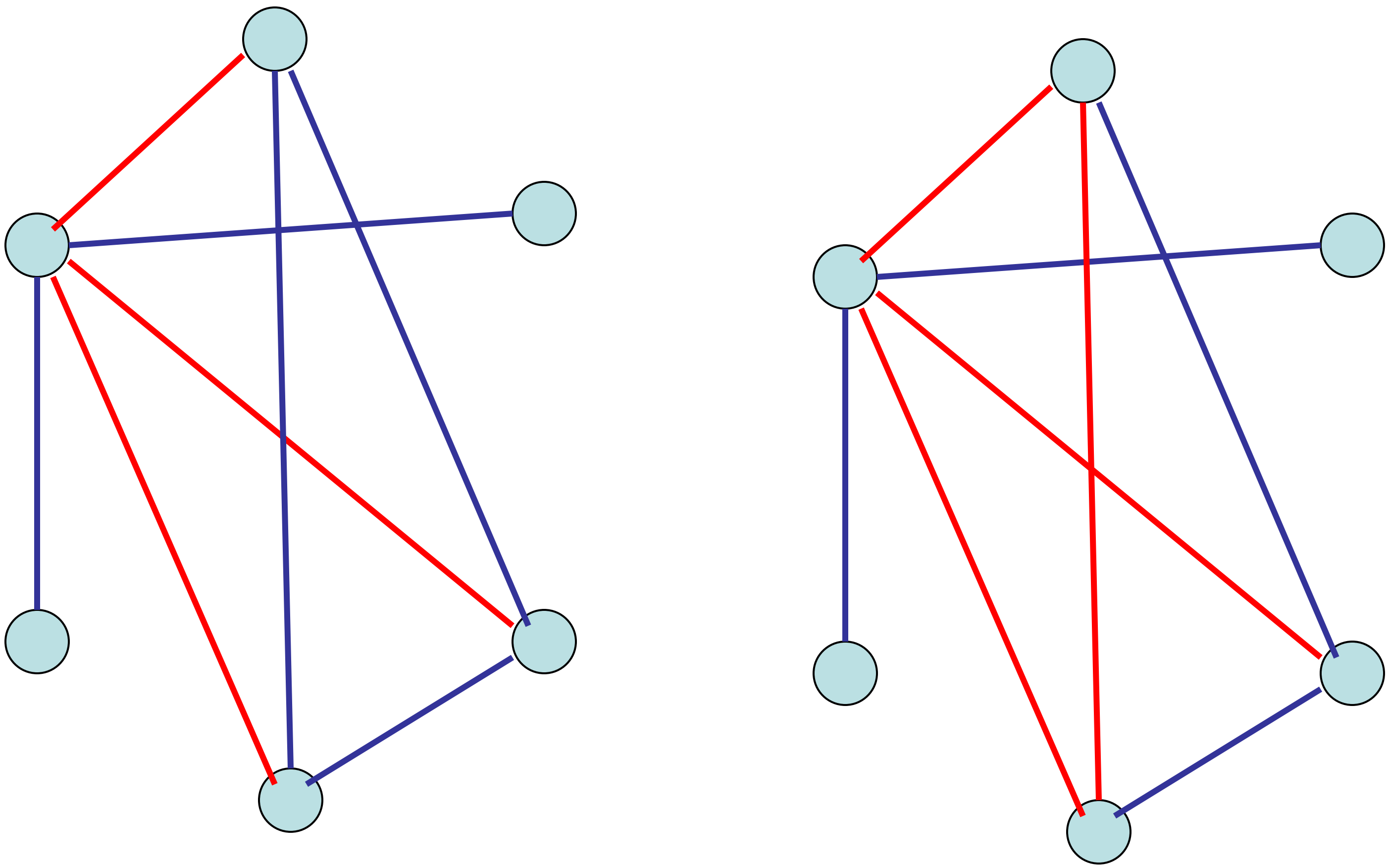
Graph Coloring

- Colour the graph: red for acquainted pairs and blue for strangers.
- Three mutually acquainted people now indicates K_3, each of whose edges are colored red.
- Likewise, three mutually unacquainted people is a blue K_3.

Assertion to be verified

- $K_6 \rightarrow K_3 K_3$
- No matter how the edges are colored with the colors red and blue, there is always a red K_3 and a blue K_3. In short there is always a mono-chromatic triangle!

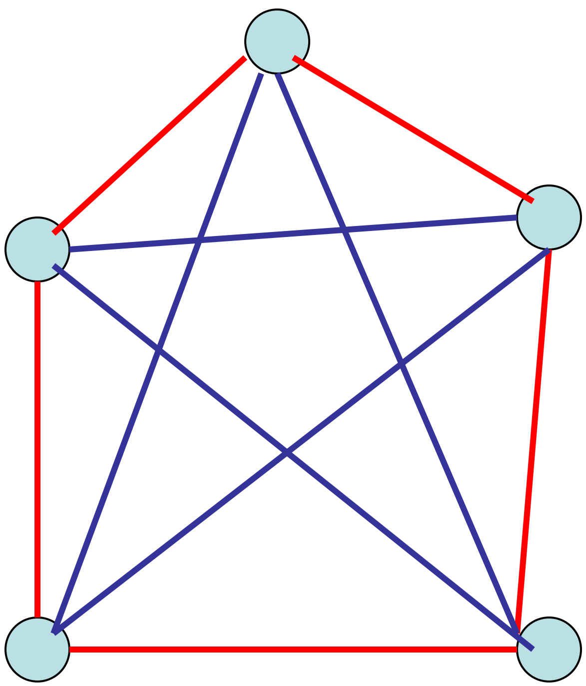
An elegant reasoning



So, we either have a K_3 of red or blue colors.

6 is the least such number

- Observe $K_5 \rightarrow K_3 K_3$



There is no monochromatic K_3.

Statement of Ramsey's Theorem

- If m≥2 and n≥2 are integers, then there is a positive integer p such that:
  - $K_p \rightarrow K_m, K_n$
  - The existence of a K_m or K_n is guaranteed, no matter how the edges of K_p are coloured.
  - p is the least such number. Thus for any integer q≥p, we have $K_q \rightarrow K_m, K_n$.
  - The ramsey number, r(m,n) is the smallest integer p, st $K_p \rightarrow K_m, K_n$
  - The existence of the number is guaranteed by Ramsey's Theorem.

Some known Ramsey's Number

- r(3,4)=9, r(4,4)=18, r(3,6)=18, r(3,5)=14, r(3,7)=23, r(4,5)=25
- The last number took 11 years of processing time on 110 desktop computers. It was discovered in 1993.
- Also, we know that r(3,8)=28 or 29. This means we know that $K_{29} \rightarrow K_3 K_8$ but $K_{27} \rightarrow K_3 K_8$ does not hold. But no one knows whether $K_{28} \rightarrow K_3 K_8$ holds.

A related problem

- There are 17 scientists who correspond to each other. They correspond about only three topics and any two treat exactly one topic. Prove that there are at least three scientists, who correspond to each other about the same subject.

Solution

- Map the problem to a graph, K_{17}. Colour the vertices red, blue or black.
- Consider a vertex v. It is connected to 16 other vertices. Since we have 3 colours, we have at least 6 edges incident on v, which are of the same color. Let it be red.
- Mark the 6 vertices as A, B, C, D, E and F.
- They form a K_6.

Solution

- If any edge of the K_6 is red, then we have a red K_3.
- Or, the K_6 is coloured with two colours, blue or black.
- So, we have either a blue or a black K_3.
- Thus, we always have a K_3 which is monochromatic.

What happens if there are 16 vertices?

- Can we do without a monochromatic triangle?
- Let us have a graph with 16 vertices.
- Label them from the set {0, a, b, c, d, a+b, a+c, a+d, b+c, b+d, c+d, a+b+c, a+b+d, a+c+d, b+c+d, a+b+c+d}
- Define: a+a=0, b+b=0, c+c=0, d+d=0
- Form three sum-free sets for the non-zero elements:
  - A_1={a,b,c,d,a+b+c+d}
  - A_2={a+b,a+c,c+d,a+b+c,b+c+d}
  - A_3={b+c,a+d,b+d,a+c+d,a+b+d}

The colouring

- Colour the edge joining x and y as x+y.
- If x+y lies in A_i, colour the edge with colour i.
- Consider, any triangle with vertices a, b and c.
- The edges are labeled as a+b, b+c and a+c.
- Since, (a+b)+(b+c)=a+c, we have the colour for the edge a+c different from that of the other two.
- The triangle is thus not monochromatic.
- A possible labeling could be using 4 binary digits.

An estimate for $r(m,n)$

- Consider the complete graph with $r(m-1,n)+r(m,n-1)$ vertices and colour the edges red and blue.
- Consider vertex v.
- v is connected to the vertices V_1 by red edges and V_2 by blue edges. Let $|V_1|=n_1$ and $|V_2|=n_2$.
- We have $n_1+n_2+1=r(m-1,n)+r(m,n-1)$.

An estimate for r(m,n)

- Thus, $n_1+n_2+1=r(m-1,n)+r(m,n-1)$.
- Either, $n_1<r(m-1,n) \Rightarrow n_2 \geq r(m,n-1)$
- This implies that V_2 has a K_m or K_n with v.
- Or, $n_1 \geq r(m-1,n) \Rightarrow V_1$ has a K_m (with v) or a K_m.
- Thus, $r(m,n) \leq r(m-1,n)+r(m,n-1)$.

Pascals Triangle Again

r(1,1)

r(1,2)   r(2,1)

r(1,3) r(2,2) r(3,1)

r(1,4) r(2,3) r(3,2) r(4,1)

r(1,5) r(2,4) r(3,3) r(4,2) r(5,1)

Suppose, we wish to have a bound for r(2,4).

Pascals Triangle Again

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Pascals Triangle Again

$$C(0,0)$$
$$C(1,0) \quad C(1,1)$$
$$C(2,0) \quad C(2,1) \quad C(2,2)$$
$$C(3,0) \quad C(3,1) \quad C(3,2) \quad C(3,3)$$
$$C(4,0) \quad C(4,1) \quad C(4,2) \quad C(4,3) \quad C(4,4)$$

$r(2,4) \le C(4,1) = 4$. $r(3,3) \le 6$.

Another Inequality

- Consider the mth row of the first and the third triangles:

r(1,m) r(2,m-1) … r(p,t) … r(m,1)

C(m-1,0) C(m-1,1) … C(m-1,m-t) … C(m-1,m-1)

Here p+t=m+1.

Thus, we have

$$\mathbf{r(p,t)} \le C(m-1,m-t)=C(p+t-2,p-1)$$

What if both the RHS terms are even?

- Let r(m-1,n)=2p,r(m,n-1)=2q and consider the graph with 2p+2q-1 vertices.
- Consider v. There can be three cases:
  - A) There are 2p-1 red and 2q-1 blue edges incident on v
  - B) There are atleast 2p red edges incident on v.
  - C) There are at least 2q blue edges incident on v.

Contd.

- Case A) cannot be true for all vertices. As then number of end points for (say) red edges is: (2p-1)(2p+2q+1) which is odd!
- So, at least for some vertices case B) or C) are true.
- Let B) be true for a vertex w. Then those 2p points has r(m-1,n). Thus there is a K_{m-1} and with w there is a K_m. Or there is a K_n.
- So, we have a graph where with less than 2p+2q vertices we have a K_m or K_n. Thus we have a strict inequality.

Some Further (easy) results

- $r(m,n)=r(n,m)$, as you can interchange the colours
- $r(2,m)=m$
  - Either some edge is coloured red (so, K_2) or all are coloured blue (so, K_m). Thus $r(2,m)\leq m$.
  - If we colour all the edges of K_{m-1} blue, we have neither a red K_2 nor a blue K_m. Thus, $r(2,m)>m-1$
  - Hence the result.

An application

- Prove that a group of 18 people will have atleast 4 mutually known people or 4 mutual strangers.
- Compute $r(4,4) \le r(3,4)+r(4,3)$
- $r(3,4) \le r(2,4)+r(3,3)=4+6=10$. In fact, $r(3,4) \le 9$. Actually $r(3,9)=9$ (Exercise prove that $r(3,4)>8$)
- Thus, $r(4,4) \le 9+9=18$. Hence the result.

Illustration of r(3,4)>8