IIT Madras
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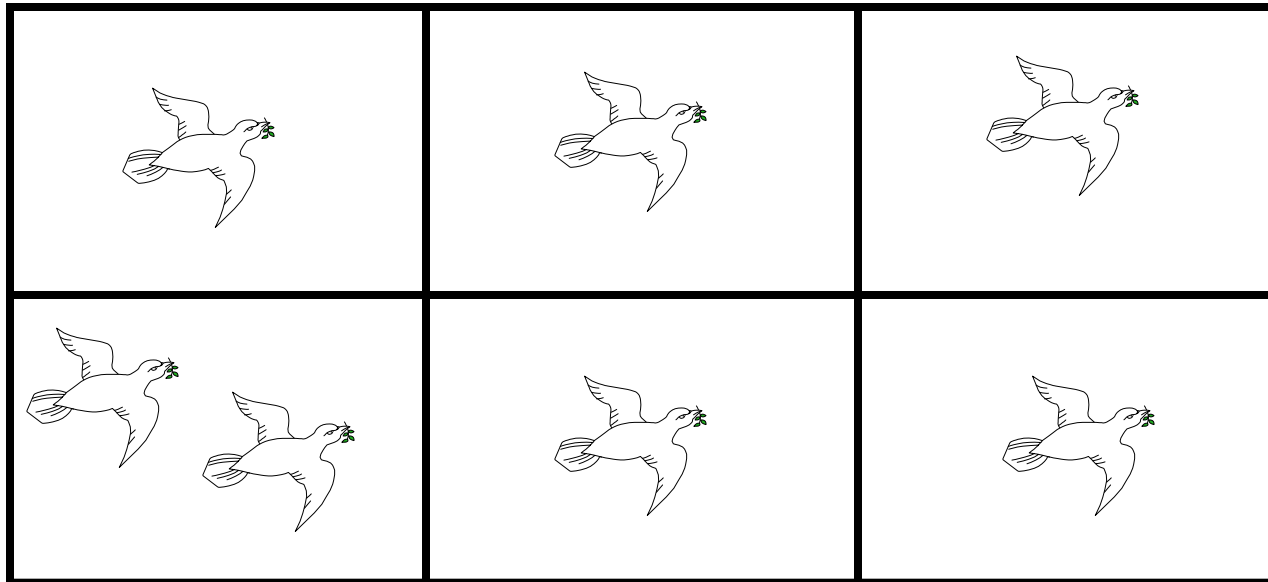
CS 210
Foundations of Computer Science

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Counting-II

Pigeonhole Principle

If n+1 or more objects (pigeons) are placed into n boxes, then there is at least one box containing two or more objects.

Note that…

- Neither the P-H principle nor its proof gives any help to find the box with more than one pigeon. Thus the principle merely guarantees the existence of a possible configuration, arrangement or phenomenon. It does not however reveal a possible construction of the arrangement.

Applications

- Application 1: Among 13 people there are at least two who have their birthdays in the same month.

- Application 2: There are n married couples. How many of the 2n people must be selected in order to guarantee that one has selected a married couple?

  n+1

Two other ways…

1. If n objects are put into n boxes and no box is empty (that is, each box has at least one element), then each box contains exactly one element

2. If n objects are put into n boxes such that each box has at the most one element, then each box has exactly an object in it.

Formally, it means…

Let X and Y be finite sets, and f: X $\to$ Y be a function from X to Y. Then :

- If X has more elements than Y, then f is not one to one.

- If X and Y have the same no of elements, and if f is onto, then f is one-one.

- If X and Y have the same no of elements, and if f is one-one, then f is onto.

Example

- Prove that from a list of integers, $a_1, a_2, \ldots, a_m$ there are always consecutive numbers whose sum is divisible by m

- Example:
  - m=6, {32,97,82,67,44,29}
  - List%6=(2,1,4,1,2,5)=> observe 1+4+1=6
  - If you make 4, 3 then we have 3+1+2=6…Its always there…
  - Check: Sum=97+82+67=246, which is divisible by 6.
  - **An interesting observation:** The numbers will be modulo m, from 0 to m-1. If we have a 0 we are done. So, assuming we don't have a 0, the remainders can be (m-1) values. So, there at least two numbers which are same.

The proof-Outline

- Equivalent list=(2,1,4,1,2,5)
- Construct sums like
  - (2)%6=2
  - (2+1)%6=3
  - (2+1+4)%6=1
  - (2+1+4+1)%6=2
  - (2+1+4+1+2)%6=4
  - (2+1+4+1+2+5)%6=3…Still there are repetitions, why?
  - Consider, sum1=2=2(modulo 6)
  -             sum2=2+1+4+1=2(modulo 6)
  - Now subtract: (sum2-sum1)=1+4+1=2-2(modulo 6)=0
  - So, 1+4+1 has to be divisible by 6
- If you have understood the answer of the why's you should get the fact that this has to always occur…
- Exercise: Write the proof formally…

Generalization

- If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain (p+1) or more pigeons, where p= $\lfloor (m-1)/n \rfloor$.

- Proof: Use method of contradiction.

- If we contradict the theorem, each hole contains p or lesser number of pigeons.

  There are n holes.

  So, total number of pigeons is less than np=n$\lfloor (m-1)/n \rfloor$≤n(m-1)/n=(m-1). But there are m pigeons. So, we have a contradiction.

Examples

- ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that there are at least two of these points, st the distance between them is less than ½ cm.

Solution

- Take an equi. triangle with sides 1 cm.

A

E

D

F

B

C
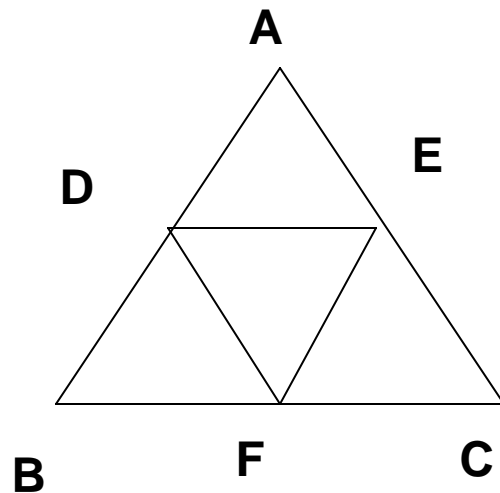
- Partition into 4 distinct groups, by constructing smaller equi triangles of sides ½ cm.
- So, one hole must have two points and their distance will be less than 0.5 cm!

Example

- A bag contains 12 pairs of socks (each pair in different colour). If a person draws the socks one by one at random, determine at most how many draws are required to get at least one pair of matched socks.

- Answer is 13.

Example

- A magnetic tape contains a collection of 5 lakh strings made up of 4 or fewer number of English letters. Can all the strings in the collection be distinct?

- There are $26^1+26^2+26^3+26^4=4,75,254$ strings possible. There are 5,00,000 strings. So, by pigeon hole principle there has to be repetition.

Example

- Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries must have at least 2045 pages.
- Pigeons=61,327, holes=30.
- Applying the generalized P-H principle:
  - At least one of the holes have more than floor((61,327-1)/30)+1 pigeons=2045 pigeons. This proves the result.

Example

- Prove that in any set of 29 persons at least five persons must have been born on the same day of the week.

- m=29, n=7.

- Thus each hole (days) must have at least floor((29-1)/7)+1=5 pigeons.

- That is at least there will be 5 persons who are born on the same day.

Examples

- Show that if any 5 numbers from 1 to 8 are chosen, then two of them will have their sum equal to 9.

- Consider the following sets:
  - $A_1=\{1,8\}$, $A_2=\{2,7\}$, $A_3=\{3,6\}$, $A_4=\{4,5\}$
  - These are the only sets with numbers from 1 to 9, st the sum is 9.
  - Thus if we choose 5 numbers, two of them will belong to one of the above 4 sets (by P-H)
  - They will add to 9.

Examples

- Prove that every set of 37 positive integers contains at least two integers that leave the same remainder upon division by 36.

- The number of remainders possible is 36 (from 0 to 36).

- So, we can construct 36 sets, A_i. It contains those numbers whose remainder is i.

- So, from 37 distinct positive integers at least two of them must belong to the same set and thus leave the same remainder.

Example

- Show that every set of seven distinct integers include two integers x and y st at least one of x+y and x-y is divisible by 10.
- Let $X=\{x_1, x_2, \ldots, x_7\}$ be a the set of the 7 distinct numbers. Let the remainder be r_i, when x_i is divided by 7.

Soln

- Consider the following six subsets:
  - $A_1 = \{x_i \in X | r_i = 0\}$
  - $A_2 = \{x_i \in X | r_i = 5\}$
  - $A_3 = \{x_i \in X | r_i = 1 \text{ or } 9\}$
  - $A_4 = \{x_i \in X | r_i = 2 \text{ or } 8\}$
  - $A_5 = \{x_i \in X | r_i = 3 \text{ or } 7\}$
  - $A_6 = \{x_i \in X | r_i = 4 \text{ or } 6\}$
- Can you complete the proof?

Example

- Prove that if 101 integers are selected from the set of S={1,2,…,200}, then at least two of them are such that one divides the other.

- Consider X={1,3,5,…,199} (the set of odd integers)

- Any number in the set S, can be represented as $2^k x_i$, where x_i is in X.

Solution

- There are 100 numbers in X. So, any 101 integers chosen will have at least two numbers who has the same x_i.

- Let the numbers be $2^k x_i$ and $2^l x_i$ ($l \geq k$). We see the smaller number divides the larger number.

Examples

- Suppose that a patient is given a prescription of 45 pills, with instructions to take at least one pill per day for 30 days. Prove that there will be a period of consecutive days during which the patient takes a total of exactly 14 pills!

- Let a_i be the number of pills the patient has taken till the end of the ith day.

Contd.

- Thus we have the following sequence:
  - $1 \leq a_1 < a_2 < \ldots < a_{30} \leq 45$

  Thus we have:
  - $1+14 \leq a_1+14 < a_2+14 < \ldots < a_{30}+14 \leq 45+14 = 59$
- Thus among all the numbers: $a_1, a_2, \ldots, a_{30}, a_1+14, a_2+14, \ldots, a_{30}+14$ are 60 numbers from 1 to 59. So, there are at two numbers which are equal. Since the a_i's are distinct, we have $a_i = a_j + 14$. Hence proved.

A more general form

- If $m=(p_1+p_2+\ldots+p_n)-n+1$ pigeons occupy n holes, $H_1,H_2,\ldots,H_n$. Prove that some H_j contains at least p_j pigeons.

- If not, then the number of pigeons is less than or equal to:

  $(p_1-1)+(p_2-1)+\ldots+(p_n-1)= =(p_1+p_2+\ldots+p_n)-n$ holes. So, we have a contradiction. Hence at least one hole, say H_i, has more than (p_i-1) pigeons. Hence proved.

Special Case

- If $p_i=k+1$, we have $m=(k+1)n-n+1=kn+1$. Hence at least one hole has more than $p_i= k+1$ pigeons.

- A bag contains many red marbles, many white marbles, and many blue marbles. What is the least number of marbles one should take out to be sure of getting at least six marbles of the same colour.

Contd.

- k+1=6=>k=5. n=3.
- Thus m≥5x3+1=16.
- Observe that if we have 15 marbles, we can have 5 red, 5 white and 5 blue marbles. Thus with 15 marbles we do not have 6 marbles of the same colour.
- Hence, 16 is indeed the answer.