Partial Orderings

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Definition

- A relation R on a set S is called a **partial ordering** if it is reflexive, antisymmetric and transitive.
- A set S together with a partial ordering R is called a **partially ordered set**, or poset, and is denoted by (S,R).
- Let X = {1,2,3,4,5,6} and P = {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (6,1), (6,4), (1,4), (6,5), (3,4), (6,2)}. Then P is partial order on X, and (X,P) is a poset.

Example

- Show that “greater than or equal” relation is a partial ordering on the set of integers.
  - $a \geq a$ for every integer a (reflexive)
  - $a \geq b$, $b \geq a$, then $a = b$ (anti-symmetric)
  - $a \geq b$, $b \geq c$, then $a \geq c$ (transitive)
- Thus $\geq$ is a partial ordering on the set of integers
- $(Z, \geq)$ is a poset.

Examples

- Similarly, the division symbol '|' is a partial ordering on the set of positive integers.
- The inclusion relation $\subseteq$ is a partial ordering on the set of P(S)
- In a poset, the notation a $\preceq$ b, indicates aRb.
- The notation, $a \prec b$ means that a $\preceq$ b, but not a=b.

Comparable and Incomparable

- The elements a and b of a poset $(S, \preceq)$ are called comparable, if either a $\preceq$ b or b $\preceq$ a. When a and b are elements of S such that neither a $\preceq$ b or b $\preceq$ a, they are called incomparable.
- In the Poset $(Z^+,|)$, are the integers 3 and 9 comparable? Yes, as $3|9 \Rightarrow 3 \preceq 9$.
- But 5 and 7 are incomparable.

Totally Ordered Sets

- If (S, $\preceq$) is a poset and every two elements of S are comparable, S is called a totally ordered set or linearly ordered set.
- It is also called a chain.
- The Poset(Z,$\leq$) is a chain.
- The Poset (Z^+,|) is not a chain.

Well Ordered Set

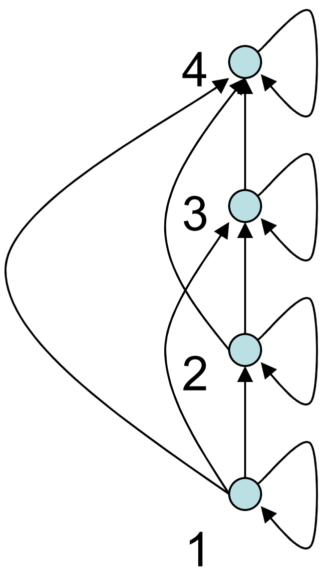
- $(S, \preceq)$ is a well ordered set if it is a poset such that $\preceq$ is a total ordering and such that every non-empty subset of S has a least element.
- Set of ordered pairs of positive integers, $Z^+ \chi Z^+$, with $(a1,a2) \preceq (b1,b2)$ if $a1 \leq b1$ or $a1=b1$and $a2 \leq b2$.
- The set Z with the usual ≤ ordering, is not well ordered.
- *Finite sets which are Totally ordered sets are well ordered (discussed in the class).*

Lexicographic Order

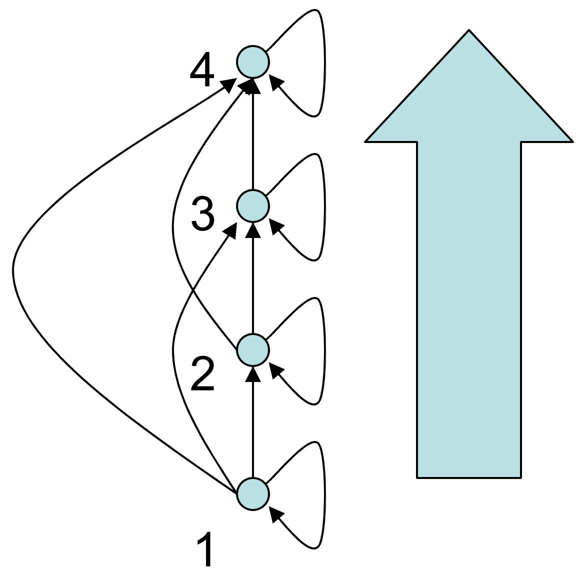
- Define an ordering on $A_1 \chi A_2$ by specifying that one pair is less than the other, if
  - The first entry of the first pair is less than the first entry of the second pair, or
  - If the first entries are equal, but the second entry of the first pair is less than the second entry of the second pair.
  - To make it partial ordering add equality to the ordering.

Hasse Diagram

- We can represent a Poset by a directed graph.



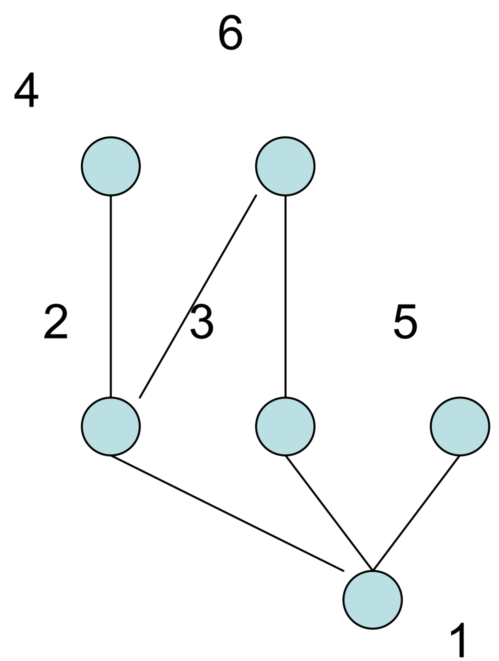
$(\{1,2,3,4\},\leq)$



All edges point upwards

1. ***Remove self loops***
2. ***Remove all edges that must be present because of transitivity.***
3. ***Also remove the arrows, as all arrows pt upwards.***

Hasse Diagram ({1,2,3,4,5,6},|)



- Hasse Diagram for the relation R represents the smallest relation R’ such that R=(R’)*

Quasi Order

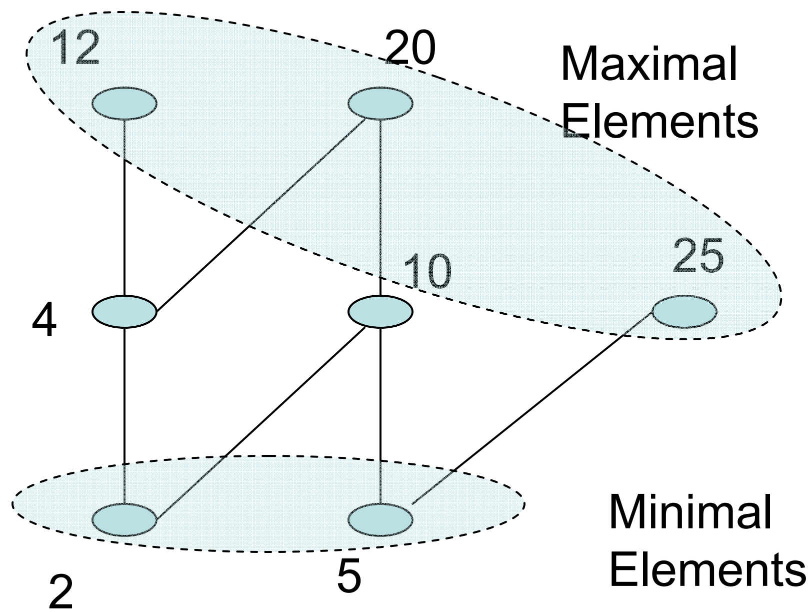
- Let R be a binary relation on A. R is a quasi order if R is transitive and irreflexive. The only distinction between a quasi order and a partial order is the equality relation.
- R is always anti-symmetric. Why?
- Example:
  - The relation < on the set of real numbers.
  - The relation “is a prerequisite” is a quasi order on any set of college courses.
  - PERT chart represents a quasi order on the collection of tasks to be performed. xRy means that task y cannot be started until task x is finished.

Maximal and Minimal Elements

- Maximal: An element a of a poset $(S, \preceq)$ is maximal if there is no element b in S, st $a \preceq b$.
- Similarly, we also have a minimal element in the poset.
- They are respectively, the "top" and the "bottom" elements in the diagram.

Example

- Which elements of the poset $(\{2,4,5,10,12,20,25\},|)$ are maximal and which are minimal?



More terms

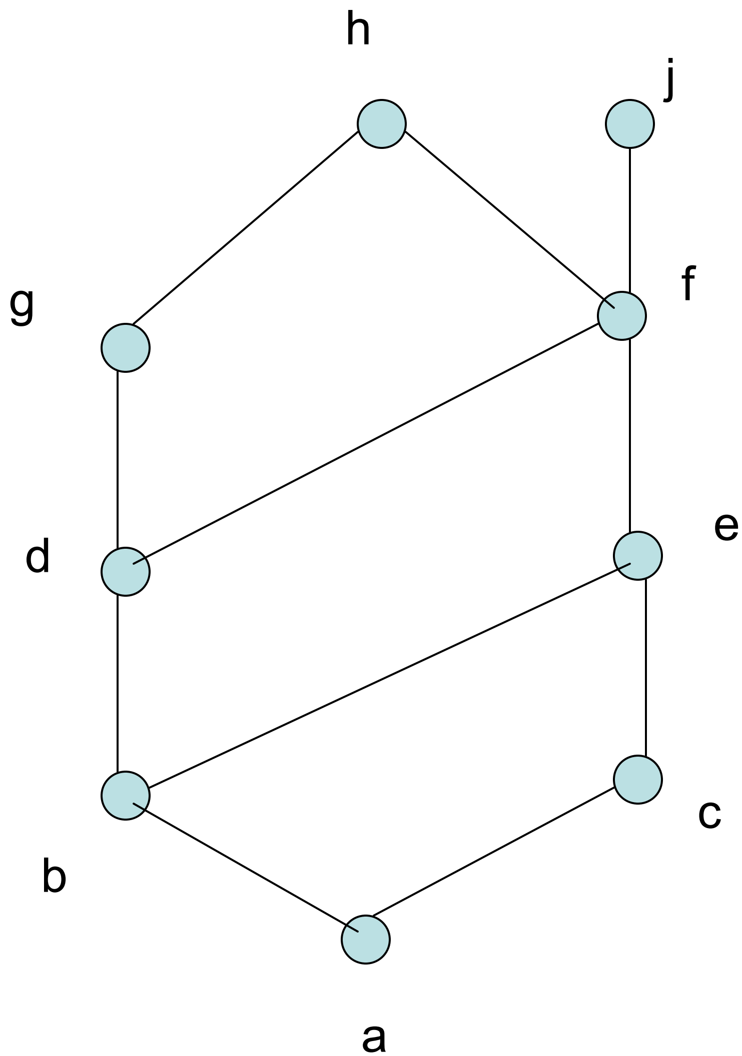
- **Greatest element:** Sometimes there is an element in a poset that is the greatest than every other elements.
- **Least element:** Sometimes there is an element which is less than all other elements in the poset.
- *The greatest and least element, when they exist are unique.*

Bounds

- Sometimes it is possible to find an element, that is greater than all the elements in a subset A of (S, $\preceq$).

  Then it is called the *upper bound* of A.
- Similarly, we have a *lower bound* of A.
- Least Upper Bound *lub(A):* Least among the upper bounds. If it exists, it is unique.
- Greatest lower Bound *glbulb(A):* Greatest among the lower bounds.

Example



- UB({a,b,c})={e,f,j,h}
- LB({a,b,c})=a
- UB({j,h})={ }
- LB({j,h})={a,b,c,d,e,f}
- UB({a,c,d,f})={h,f,j}
- LB ({a,c,d,f})={a}
- glb({b,d,g})=max({a,b})=b
- lub({b,d,g})=min({g,h})=g

More Examples

- Find the glb and lub of the sets {3,9,12} and {1,2,4,5,10} if they exist in the poset $(Z^+,|)$.
- glb=3
- lub=36.

Lattices

- **Lattices:** A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound.
- They are very useful as models of information flow and Boolean algebra.

Which one are lattices?

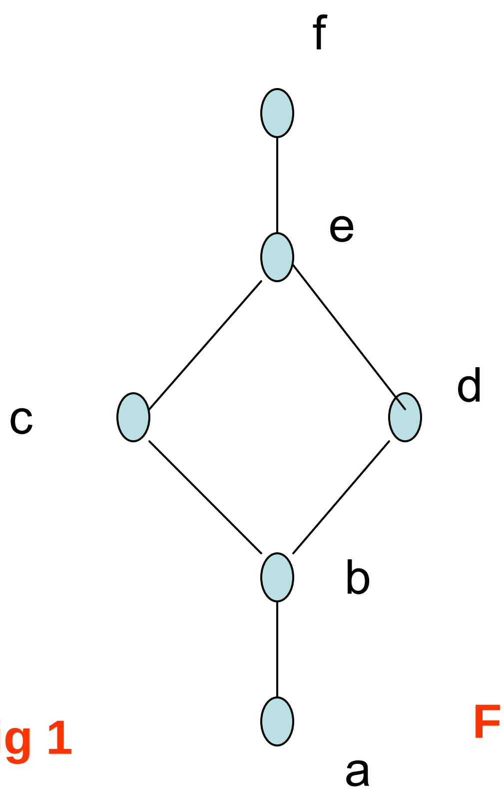


Fig 1

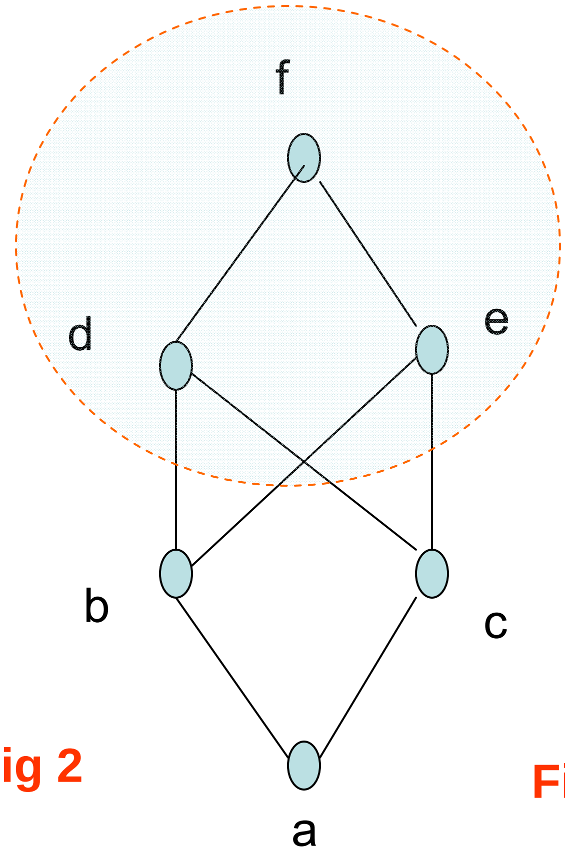


Fig 2

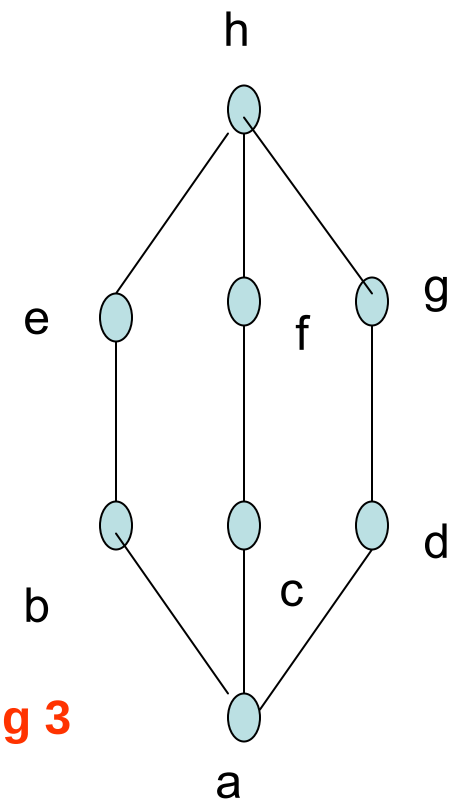


Fig 3

- **Fig 1 and 3 are lattices. Fig 2 is not because, {b,c} has no lub**
- **However, it has a glb={a}**

Topological Sorting

- A total ordering is said to be ***compatible*** with the partial ordering R if $a \preceq b$ whenever aRb.
- aRb => $a \preceq b$ (Partial => Total)
- Constructing a compatible total ordering from a partial ordering is called ***topological sorting.***

Theorem

- Every finite non-empty poset (S, $\preceq$) has a minimal element.
- Proof is left as an exercise. It follows from the fact that the set is finite and so our search for an minimal element should terminate at one point.

Algorithm

procedure topological sort(S:finite poset)

k=1

while S≠Ø

begin

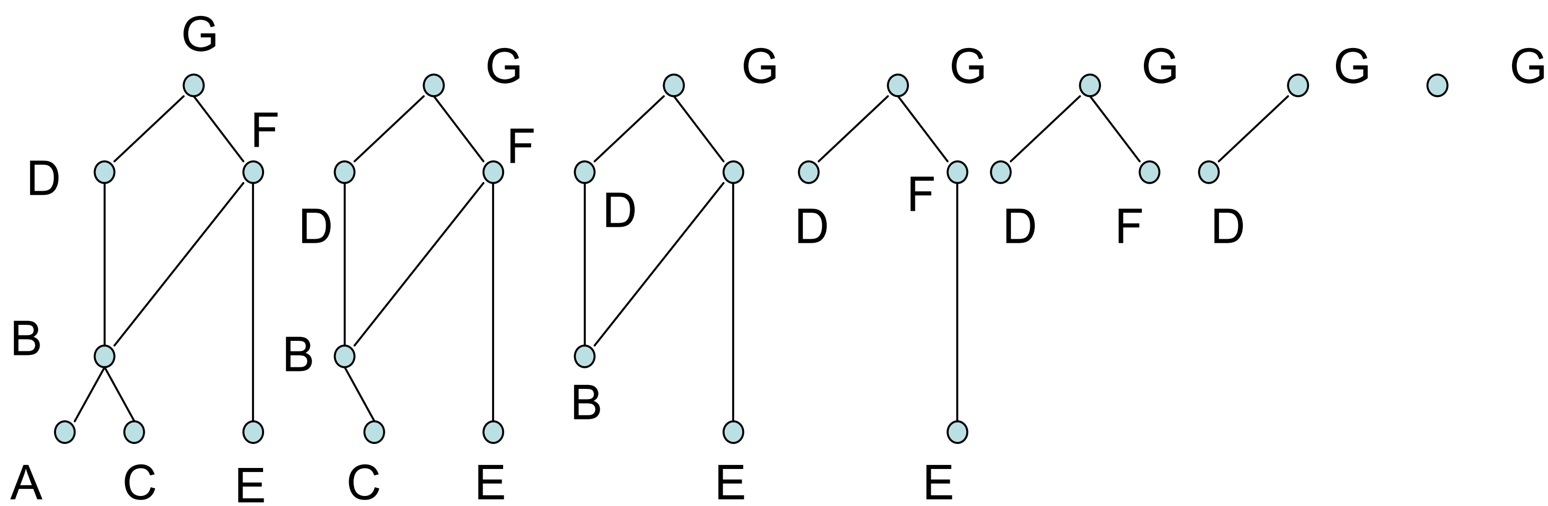
a_k=a minimal element of S{such an element exists by lemma 1}

S=S-{a_k}

k=k+1

end{$a_1,a_2,\ldots,a_n$ is a compatible total ordering of S}

Example of a Topological Sort



- A $\preceq$ C $\preceq$ B $\preceq$ E $\preceq$ F $\preceq$ D $\preceq$ G

Minimal Element Chosen