The Principle of Inclusion-Exclusion

Debdeep Mukhopadhyay
IIT Madras

Cardinality

- S is a finite set
- Number of elements in the set is called the cardinality of the set.
- It is denoted by $|S|$
- **Basic Principle:**
  - $|A \cup B| = |A| + |B| - |A \cap B|$
  - That is for determining the number of elements in the union of A and B, we ***include*** all the elements in A and B, but ***exclude*** all elements common to A and B.

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$$
\begin{aligned}
|\overline{A} \cap \overline{B}| &= (\overline{A \cup B}) \\
&= |S| - |A \cup B| \\
&= |S| - |A| - |B| + |A \cap B|
\end{aligned}
$$

Both the formulae are equivalent and are referred to as the Addition Principle or the Principle of Inclusion Exclusion.

Generalization

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| + \\ \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n|$$

Note that any element x which belongs to $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ should also be there only one time in the right side of the equation.

Let us count the number of times x occurs in the right hand side. Let x belong to m sets out of the A_i's. Thus the number of times x occurs in the RHS is:

$$m - C(m,2) + C(m,3) + \ldots + (-1)^{m-1} C(m,m) = 1 - \{1 - m + C(m,2) - C(m,3) + \ldots + (-1)^m C(m,m)\}$$

$$= 1 - (1 + (-1))^m = 1$$

This proves the result.

Corollary

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \ldots + (-1)^n |A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n|$$

Suppose, A_1 represents the set of all elements of S, which satisfies condition c_1, A_2 represents the set of all those elements of S which satisfies condition c_2 and so on.
Then we can rewrite the above equations as follows:

$$N(c_1 \text{ or } c_2 \text{ or } \ldots \text{ or } c_n) = \Sigma N(c_i) - \Sigma N(c_i c_j) + \Sigma N(c_i c_j c_k) - \ldots + (-1)^{n-1} N(c_1 c_2 \ldots c_n)$$
$$= S_1 - S_2 + \ldots + (-1)^{n-1} S_n$$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \ldots + (-1)^n S_n$$

Further Generalizations

- *Number of elements in S which satisfies at least m of the n conditions:*

$L_m=S_m-C(m,m-1)S_{m+1}+C(m+1,m-1)S_{m+2}+\ldots+(-1)^{n-m}C(n-1,m-1)S_n$

- *Number of elements in S which satisfies exactly m of the n conditions:*

$E_m=S_m-C(m+1,1)S_{m+1}+C(m+2,2)S_{m+2}+\ldots+(-1)^{n-m}C(n,n-m)S_n$

Example

- Out of 30 students in a hostel, 15 study History, 8 study Economics and 6 study Geography. It is known that 3 study all the subjects. Show that 7 or more study none of the subjects.
- $|A|=15$, $|B|=8$, $|C|=6$, $|A \cap B \cap C|=3$

$$
\begin{aligned}
|\overline{A} \cap \overline{B} \cap \overline{C}| &= |S| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \\
&= |S| - S_1 + S_2 - S_3 = 30 - 29 + S_2 - 3 = S_2 - 2
\end{aligned}
$$

But, $(A \cap B \cap C)$ is a subset of $A \cap B$. Thus, $|A \cap B| \le |A \cap B \cap C|$.
Thus $S_2 \le 3|A \cap B \cap C| = 9 \rightarrow S_2 - 2 \le 7$

Example

- Determine the number of positive integers n st $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.
- Define S={1,2,3,…,100}
- Define |A|=no of multiples of 2 in S=$\lfloor 100/2 \rfloor = 50$
- Define |B|=no of multiples of 3 in S=$\lfloor 100/3 \rfloor = 33$
- Define |C|=no of multiples of 5 in S=$\lfloor 100/5 \rfloor = 20$

$$|\overline{A}\cap\overline{B}\cap\overline{C}| = |S| - |A| - |B| - |C| + |A\cap B| + |A\cap C| + |B\cap C| - |A\cap B\cap C|$$
$$= 100 - (50+33+20) + (16+10+6) - 3 = 26$$

Example

- Find the number of nonnegative integer solutions of the equation:

$$x_1+x_2+x_3+x_4=18$$

  under the condition $x_i \leq 7$ for i=1,2,3,4

- Let S denote the set of all non-negative integral solutions of the given equation. The number of such solutions is C(18+4-1,4-1)=C(21,3)=> |S|=C(21,3)

Contd.

- Define subsets:
  - $A_1=\{(x_1,x_2,x_3,x_4)\in S|x_1>7\}$
  - $A_2=\{(x_1,x_2,x_3,x_4)\in S|x_2>7\}$
  - $A_3=\{(x_1,x_2,x_3,x_4)\in S|x_3>7\}$
  - $A_4=\{(x_1,x_2,x_3,x_4)\in S|x_4>7\}$
  - The required number of solns is $|\overline{A_1}\cap\overline{A_2}\cap\overline{A_3}\cap\overline{A_4}|$

So, the next question is how to we obtain $|A_1|$?

Set, $y_1=x_1-8$. Thus the eqn becomes $y_1+x_2+x_3+x_4=10$.

And the number of non-negative integral solutions are C(10+4-1,4-1)=C(13,3). This is the value of $|A_1|$. By symmetry, $|A_1|=|A_2|=|A_3|=|A_4|$.

Contd.

The next question is how to we obtain $|A_1 \cap A_2|$?

Set, $y_1=x_1-8$ and $y_2=x_2-8$. Then the eqn becomes $y_1+y_2+x_3+x_4=2$.

The number of non-negative integral solutions is C(2+4-1,4-1)=C(5,3)

This is the value of $|A_1 \cap A_2|$.

By symmetry, $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4|$

Observing from the original equation, more than 2 variables cannot be more than 7.

Thus from,

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| &= |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \\ &\quad \ldots + |A_1 \cap A_2 \cap A_3 \cap A_4| \\ &= C(21,3) - C(4,1)C(13,3) + C(4,2)C(5,2) - 0 + 0 = 366\end{aligned}$$

Example

- In how many ways 5 a's, 4 b's and 3 c's can be arranged so that all the identical letters are not in a single block?
- If S is the set of all permutations, $|S|=12!/(5!4!3!)$
- Let A1 be the set of permutations of the letters, where the 5 a's are in a single block: $|A1|=8!/(4!3!)$
- Similarly if A2 is the set of arrangements such that the 4 b's are together and A3 is the set of arrangements such that all the 3 c's are in a single block: $|A2|=9!/(5!3!)$, $|A3|=10!/(5!4!)$
- Rest is left as an exercise.

Example

- In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns CAR, DOG, PUN or BYTE occurs?

Example

- In a certain area of the country side, there are 5 villages. You are to devise a system of roads so that after the system is completed, no village will be isolated. In how many ways can we do this?