Growth of Functions

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Asymptotic Performance

- Exact running time of an algorithm is not always required:
  - When the input size of a problem is very large. Like, in the insertion sort example if the number of elements we had to sort are very large.
  - Then the multiplicative constants and the lower order terms can be neglected.
- ***How the running time of an algorithm increases when the input increases unbounded ?***

Growth of Functions

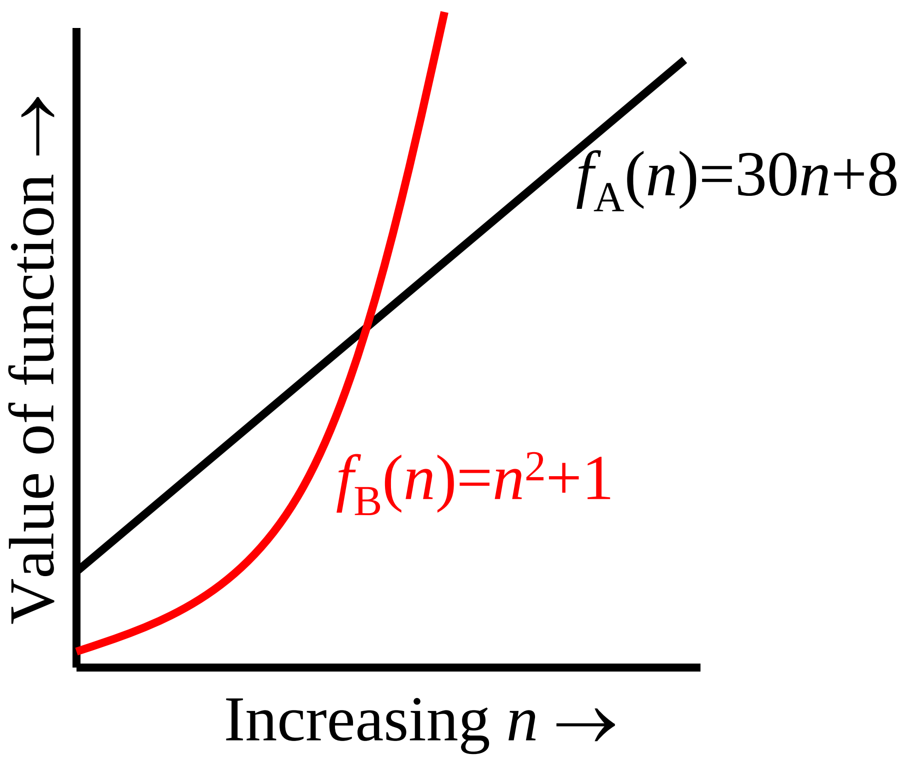
- For functions over numbers, we often need to know a rough measure of *how fast a function grows*.
- If $f(x)$ is *faster growing* than $g(x)$, then $f(x)$ always eventually becomes larger than $g(x)$ *in the limit* (for large enough values of x).
- Useful in engineering for showing that one design *scales* better or worse than another.

Growth of Functions

- Suppose you are designing a web site to process user data (*e.g.*, financial records).
- Suppose database program A takes $f_A(n)=30n+8$ microseconds to process any n records, while program B takes $f_B(n)=n^2+1$ microseconds to process the n records.
- Which program do you choose, knowing you'll want to support millions of users?

Visualizing Growth of Functions

- On a graph, as you go to the right, a faster growing function eventually becomes larger...



Definition: *O*(*g*), *at most order g*

Let g be any function $\mathbf{R}\rightarrow\mathbf{R}$.

- Define "*at most order g*", written O(g), to be:

  $$\{f:\mathbf{R}\rightarrow\mathbf{R} \mid \exists +ve\ c,k:\ \forall x>k:\ 0 \le f(x) \le cg(x)\ \}$$

  – "Beyond some point k, function f is at most a constant c times g (*i.e.,* proportional to g)."

  – We are dealing with asymptotically nonnegative elements of the set

- "*f* is *at most order g*", or "*f* is O(*g*)", or **"*f*=O(*g*)" all just mean that $f\in$O(g).**

Points about the definition

- Note that *f* is O(*g*) so long as *any* values of *c* and *k* exist that satisfy the definition.
- But: The particular *c*, *k*, values that make the statement true are *not* unique: **Any larger value of *c* and/or *k* will also work.**
- You are **not** required to find the smallest *c* and *k* values that work. (Indeed, in some cases, there may be no smallest values!)

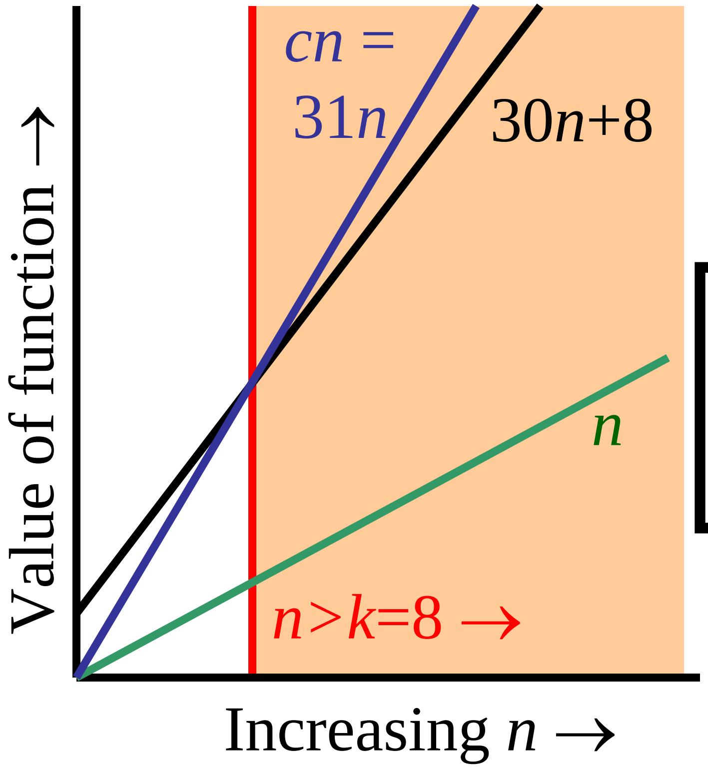
However, you should **prove** that the values you choose do work.

"Big-O" Proof Examples

- Show that $30n+8$ is $O(n)$.
  - Show $\exists c,k$: $\forall n>k$: $30n+8 \leq cn$.
    - Let $c=31$, $k=8$. Assume $n>k=8$. Then $cn = 31n = 30n + n > 30n+8$, so $30n+8 < cn$.
- Show that n^2+1 is $O(n^2)$.
  - Show $\exists c,k$: $\forall n>k$: $n^2+1 \leq cn^2$.
    - Let $c=2$, $k=1$. Assume $n>1$. Then $cn^2 = 2n^2 = n^2+n^2 > n^2+1$, or $n^2+1< cn^2$.

Big-O example, graphically

- Note $30n+8$ isn't less than n *anywhere* ($n>0$).
- It isn't even less than $31n$ *everywhere*.
- But it *is* less than $31n$ <u>everywhere to the right of $n=8$</u>.



$$30n+8 \in O(n)$$

Definition: $\Theta(g)$, *exactly order g*

- If $f \in O(g)$ and $g \in O(f)$ then we say "*g and f are of the same order*" or "*f is (exactly or tightly) order g*" and write $f \in \Theta(g)$.
- Another equivalent definition:
  $\Theta(g) \equiv \{f:\mathbf{R} \rightarrow \mathbf{R} \mid$
  $\exists \text{+ve } c_1 c_2 k\ \forall x > k:\ 0 \leq c_1 g(x) \leq f(x) \leq c_2 g(x)\}$
- "Everywhere beyond some point k, $f(x)$ lies in between two multiples of $g(x)$."

Definition: $\Omega(g)$, at least *order g*

Let g be any function $\mathbf{R}\rightarrow\mathbf{R}$.

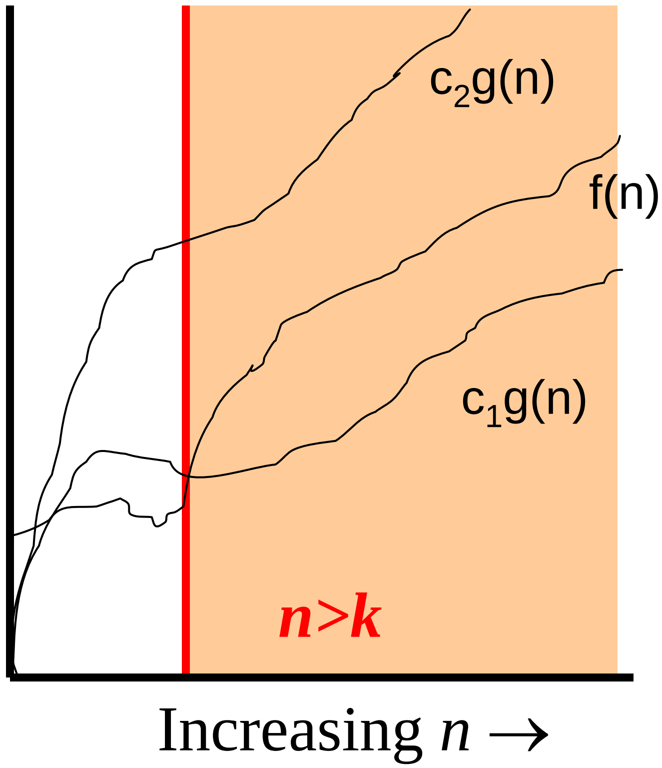
- Define "*at most order g*", written $O(g)$, to be:

$\{f:R\rightarrow R \mid \exists$+ve c,k: $\forall x>k$: $f(x) \geq cg(x) \geq 0\}$

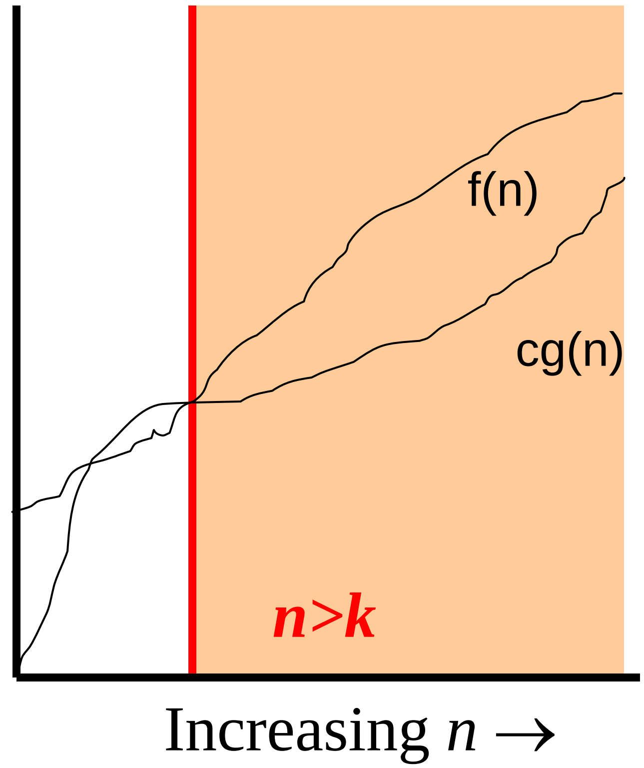
  - "Beyond some point k, function f is at least a constant c times g (*i.e.,* proportional to g)."

- "*f* is *at least order g*", or "*f* is $\mathbf{\Omega}(g)$", or **"$f=\Omega(g)$" all just mean that $f\in\Omega(g)$.**

Graphical Representation



***f(n)*=Θ(*g(n)*)**



f(n)=Ω(g(n))

An Example of Tight Bound (Θ)

- Prove $f(n)= \frac{1}{2} n^2-3n = \mathbf{\Theta(n^2)}$
- **In order to prove this we require constants: c_1 and c_2 s.t. :**
  - $c_1 n^2 \leq \frac{1}{2} n^2-3n \leq c_2 n^2$, **for all** $n \geq n_0$
  - $c_1 \leq \frac{1}{2} - 3/n \leq c_2$, **for all** $n \geq n_0$

n	1	2	3	4	5	6	7	8
f(n)	-5/2	-1	-1/2	-1/4	-1/10	0	1/14	>1/14

Set $n_0=7$, $c_1=1/14$, $c_2=1/2$.
<u>It is not important to have an unique value, what is important that one set of values exist.</u>

For this class

- We shall be using the O-notation in the class frequently
- Point to be kept in mind: *If running time is $O(n^2)$=> there is a function $f(n)$ that is $O(n^2)$ s.t. for any value of $n \geq n_0$, no matter what particular input of size n is chosen, the running time for that input is bounded from above by the value $f(n)$.*

Next Day Recurrences