Functions

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Functions

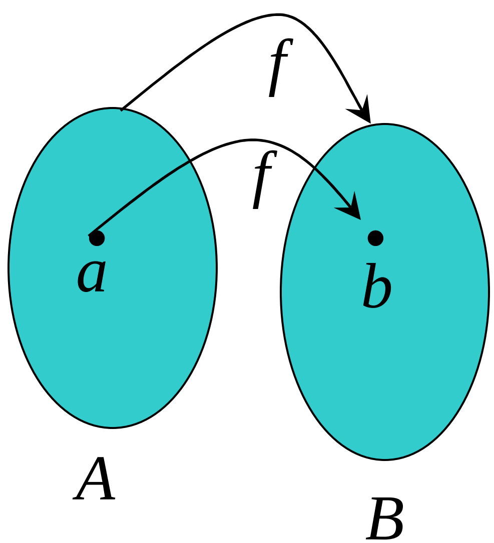
- From calculus, you are familiar with the concept of a real-valued function f, which assigns to each number $x \in \mathbf{R}$ a particular value $y=f(x)$, where $y \in \mathbf{R}$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *map*.)

Function: Formal Definition

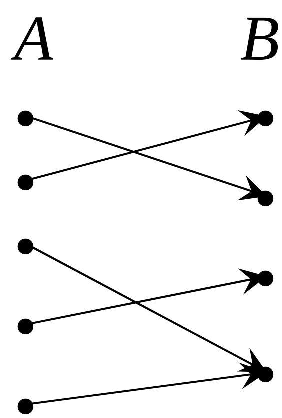
- For any sets A, B, we say that a *function f from (or "mapping") A to B* ($f:A\rightarrow B$) is a particular assignment of exactly one element $f(x)\in B$ to each element $x\in A$.
- Some further generalizations of this idea:
  - A *partial* (non-*total*) function f assigns *zero or one* elements of B to each element $x\in A$.
  - Functions of n arguments; relations

Graphical Representations

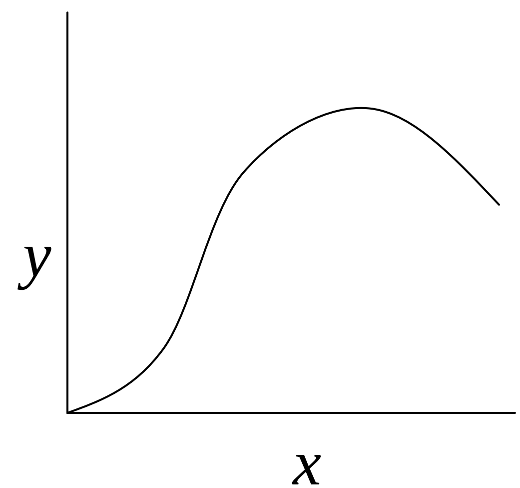
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Functions We've Seen So Far

- A *proposition* can be viewed as a function from "situations" to truth values {**T**,**F**}
- A *propositional operator* can be viewed as a function from *ordered pairs* of truth values to truth values: *e.g.*, $\vee((\mathbf{F},\mathbf{T})) = \mathbf{T}$.

Another example: $\rightarrow((\mathbf{T},\mathbf{F})) = \mathbf{F}$.

More functions

- A *predicate* can be viewed as a function from *objects* to *propositions* (or truth values): $P :\equiv$ "is 7 feet tall"; P(Mike) = "Mike is 7 feet tall." = **False**.
- A *bit string B of length n* can be viewed as a function from the numbers $\{1,\ldots,n\}$ (bit *positions*) to the *bits* {0,1}. *E.g.*, B=101 → $B(3)=1$.

Still More Functions

- A *set* S over universe U can be viewed as a function from the elements of U to {**T**, **F**}, saying for each element of U whether it is in S. S={3}$\Rightarrow$ S(0)=**F**, S(3)=**T**.
- A *set operator* such as $\cap$,$\cup$, $\overline{}$ can be viewed as a function from pairs of sets to sets.
  - Example: $\cap((\{1,3\},\{3,4\})) = \{3\}$

Notations

- Sometimes we write Y^X to denote the set F of *all* possible functions $f{:}X \rightarrow Y$.
- This notation is especially appropriate, because for finite X, Y, we have $|F| = |Y|^{|X|}$.
- If we use representations $\mathbf{F} \equiv \mathbf{0}$, $\mathbf{T} \equiv \mathbf{1}$, $\mathbf{2} :\equiv \{\mathbf{0},\mathbf{1}\} = \{\mathbf{F},\mathbf{T}\}$, then a subset $T \subseteq S$ is just a function from S to $\mathbf{2}$, so the power set of S (set of all such fns.) is $\mathbf{2}^S$ in this notation.

Some Function Terminology

- If it is written that f:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then we say:
  - – A is the *domain* of f.
  - – B is the *codomain* of f.
  - – b is the *image* of a under f.
  - – a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - – The *range* $R \subseteq B$ of f is $R=\{b \mid \exists a\; f(a)=b\}$.

We also say the *signature* of f is $A \rightarrow B$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know *f*'s codomain is: {S,A,B,C,D,E}, and its range is unknown!.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}, but its codomain is still {A,B,C,D,E}!.

Function Operator Example

- $+$,$\times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* f,g:**R**→**R**:
  - $(f + g)$:**R**→**R**, where $(f + g)(x) = f(x) + g(x)$
  - $(f \times g)$:**R**→**R**, where $(f \times g)(x) = f(x) \times g(x)$

Function Composition Operator

Note both are B

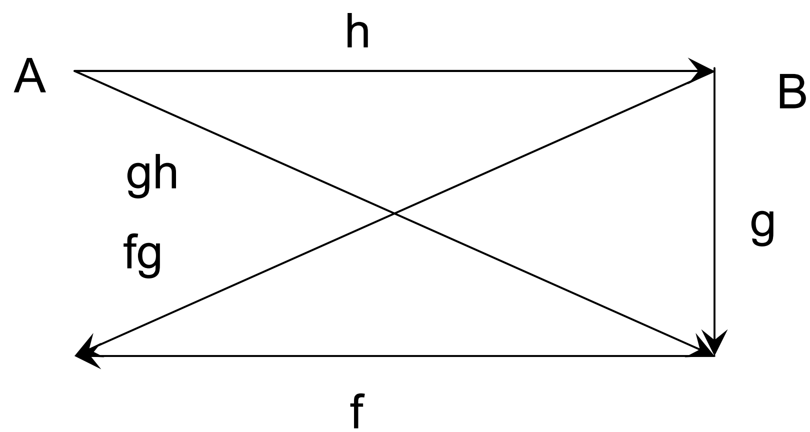
- For functions $g{:}A\rightarrow B$ and $f{:}B\rightarrow C$, there is a special operator called *compose* ("$\circ$").
  - It composes (creates) a new function out of f and g by applying f to the result of applying g.
  - We say $(f\circ g){:}A\rightarrow C$, where $(f\circ g)(a) :\equiv f(g(a))$.
  - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that $\circ$ (like Cartesian $\times$, but unlike $+,\wedge,\cup$) is non-commuting. (Generally, $f\circ g \neq g\circ f$.)

Commutative Diagram

$$
\begin{tikzcd}
A \arrow[rr, "fg"] \arrow[dr, "g"'] & & C \\
& B \arrow[ur, "f"'] &
\end{tikzcd}
$$

- Each commutation (path) in the graph represents a composite of functions which appear as labels in the path.

Composition is Associative



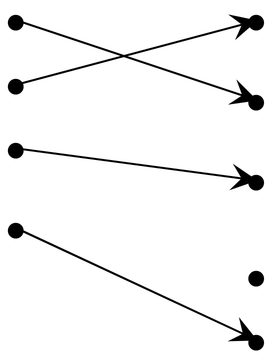
- f(gh)=(fg)h (Associative)

One-to-One Functions

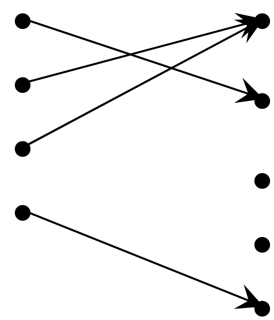
- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
  - Formally: given $f:A \rightarrow B$, "x is injective" $:\equiv$ $(\neg\exists x,y{:}\ x \neq y \wedge f(x)=f(y))$.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain?

One-to-One Illustration

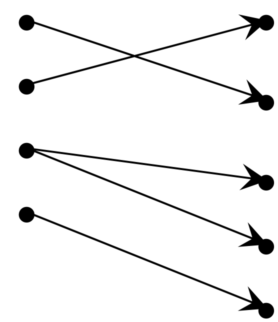
- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

Sufficient Conditions for 1-1ness

- For functions *f* over numbers, we say:
  - *f* is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all *x*,*y* in domain;
  - *f* is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all *x*,*y* in domain;
- If *f* is either strictly increasing or strictly decreasing, then *f* is one-to-one. *E.g.* x^3
  - *Converse is not necessarily true. E.g.* $1/x$
  - *Here the domain is the set of integers*

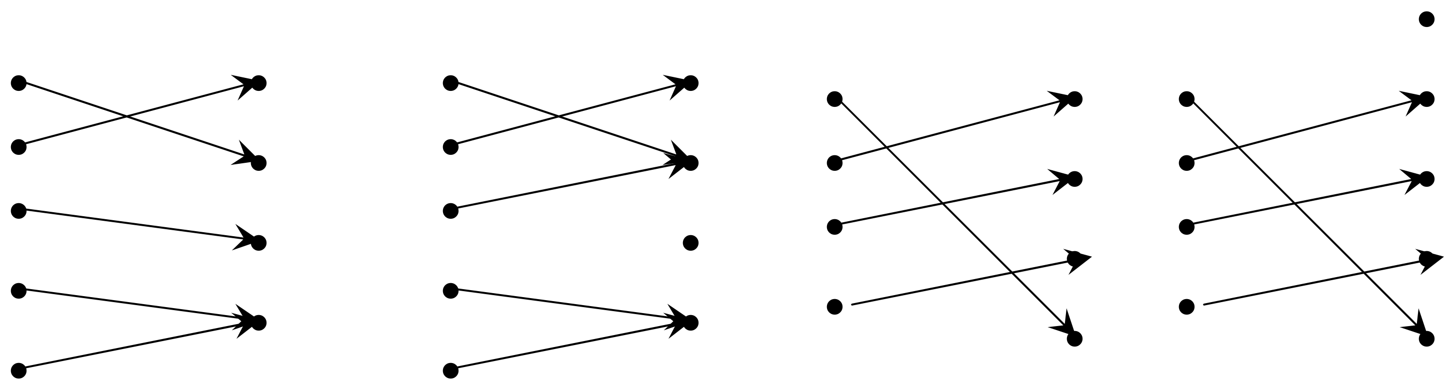
In general, the definitions hold when the domain and codomain of a function are linearly ordered.

Onto (Surjective) Functions

- A function $f{:}A\rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b\in B, \exists a\in A{:}\ f(a)=b$).
- Think: An *onto* function maps the set A onto (over, covering) the *entirety* of the set B, not just over a piece of it.
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are, or are not, *onto* their codomains:



Onto (but not 1-1) | Not Onto (or 1-1) | Both 1-1 and onto | 1-1 but not onto

Bijections

- A function *f* is said to be *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is
<u>both</u> one-to-one <u>and</u> onto.
- For bijections $f{:}A \rightarrow B$, there exists an *inverse of f*, written $f^{-1}{:}B \rightarrow A$, which is the unique function such that $f^{-1} \circ f = I_A$
  - (where I_A is the identity function on *A*)
- A Permutation on set A is a bijective function on A.
- Permutation is closed under composition.
  - Take two bijective functions and compose them. You have another permutation.

Some properties

- If f and g are surjective, then fg is surjective
- If f and g are injective, then fg is injective
- If f and g are bijective, then fg is bijective.

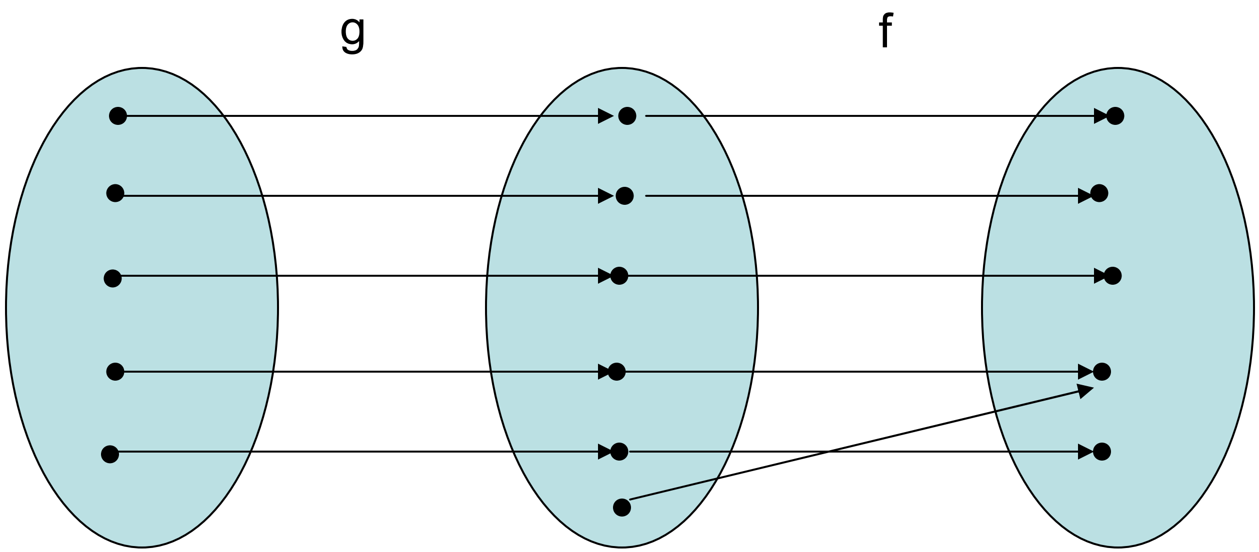
Some more properties

- If fg is surjective, then f is surjective
- If fg is injective, then g is injective
- If fg is bijective, then f is surjective and g is injective.

Explanation of 2nd Point of last slide

- Define g: $A \rightarrow B$, f: $B \rightarrow C$, fg: $A \rightarrow C$
- Let $a \neq b$ belong to A. Since fg is injective, we have $f(g(a)) \neq f(g(b))$.
- Now, $g(a) \neq g(b)$, as otherwise we have a contradiction.
- Hence, g is injective.
- But is f also injective? As, if $g(a)=k_1$ and $g(b)=k_2$, and $k_1 \neq k_2 \Rightarrow f(k_1) \neq f(k_2)$.

Pictorial Representation



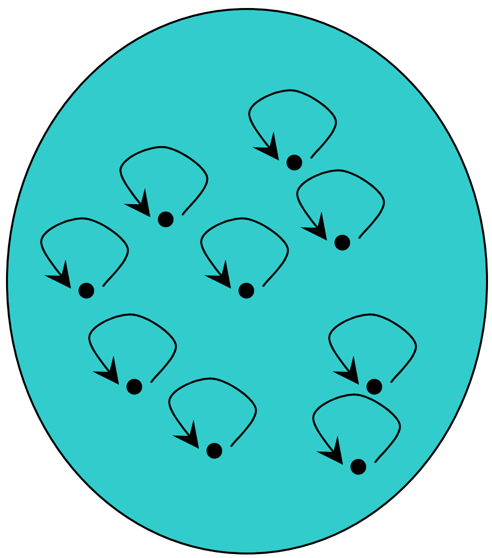
- fg is injective
- Hence, g is injective.
- But f is not injective.

The Identity Function

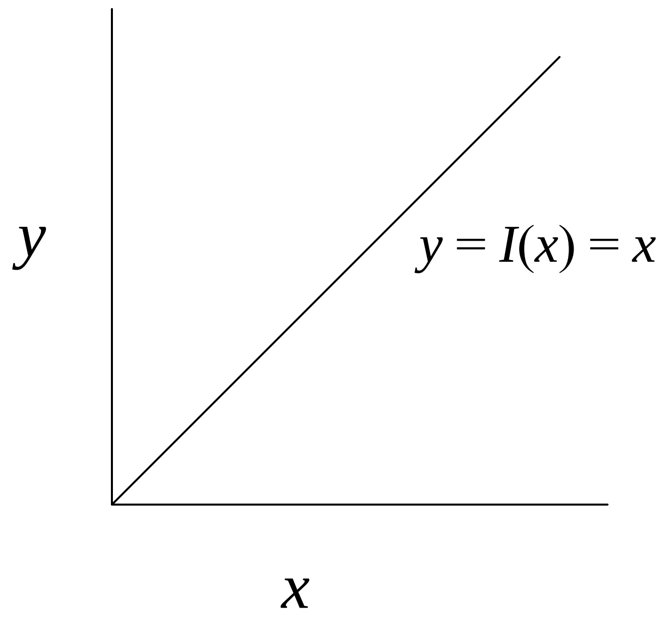
- For any domain A, the *identity function* $I:A\rightarrow A$ (variously written, I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a\in A\colon I(a)=a$.
- Some identity functions you've seen:
  - $+$ing 0, $\cdot$ing by 1, $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\varnothing$, $\cap$ing with U.
- Note that the identity function is always both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



Application

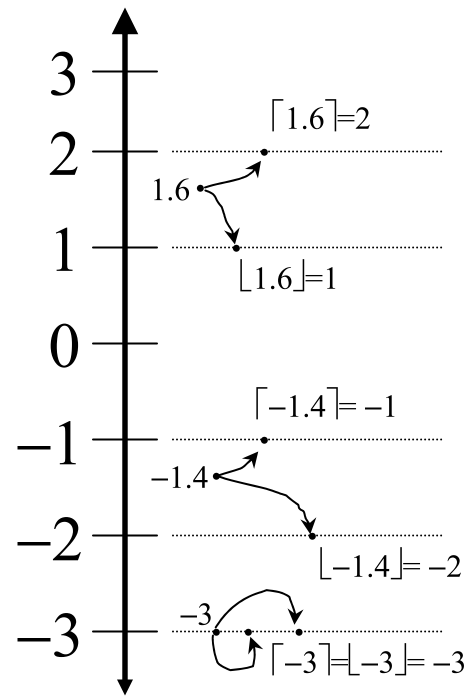
- In a computer, the function mapping the machine's state at clock cycle #*t* to its state at clock cycle #*t*+1 is called the computer's *transition function*.
- If the transition function happens to be reversible (a bijection), then the computer's operation, in theory, requires no energy expenditure.

A Couple of Key Functions

- In discrete math, we will frequently use the following two functions over real numbers:
  - The *floor* function $\lfloor \cdot \rfloor : \mathbf{R} \rightarrow \mathbf{Z}$, where $\lfloor x \rfloor$ ("floor of *x*") means the largest (most positive) integer $\leq x$. *I.e.*, $\lfloor x \rfloor :\equiv \max(\{i \in \mathbf{Z} \mid i \leq x\})$.
  - The *ceiling* function $\lceil \cdot \rceil : \mathbf{R} \rightarrow \mathbf{Z}$, where $\lceil x \rceil$ ("ceiling of *x*") means the smallest (most negative) integer $\geq x$. $\lfloor x \rfloor :\equiv \min(\{i \in \mathbf{Z} \mid i \geq x\})$

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ & $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.



Plots with floor/ceiling

- Note that for $f(x)=\lfloor x \rfloor$, the graph of f includes the point (a, 0) for all values of a such that $a\geq 0$ and $a<1$, but not for the value $a=1$.
- We say that the set of points (a,0) that is in f does not include its *limit* or *boundary* point (a,1).
  - Sets that do not include all of their limit points are generally called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$: