Derangements

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The basic problem

- A permutation of n distinct objects in which none of the objects is in its natural or original place is called derangement.
- We represent the number of n distinct objects by U_n.
- Thus $U_1=0$.
- Can we derive an explicit formula for U_n?

A recursive proof

- Let n-letters be placed in n-Envelopes in wrong way in U_n ways.
- With respect to letter L1, Un can be divided into two parts:
  - L1 and another Letter Lk are cross-placed into Ek and E1 respectively, which can done in (n – 1) ways, where rest of the other (n – 2) Letters can be inserted into remaining (n – 2) Envelopes in a wrong way by U_{n-2} ways. Total number of such cases = $(n – 1)\, U_{n-2}$

Contd.

- L1 can be placed in any Envelope (other than E1) and no corresponding cross-placed is involved. In this case, L1 can be placed in $(n - 1)$ Envelopes and the remaining $(n - 1)$ Letters can be inserted into remaining $(n - 1)$ Envelopes by U_{n-1} ways.
- Total number of such cases =
  $(n - 1)\, U_{n-1}$

Obtaining a recursive relation

- Thus we have:
  - $U_n = (n-1)\, U_{n-2} + (n-1)\, U_{n-1}$

  Or,
  - $U_n - n\, U_{n-1} = -\,[U_{n-1} - (n-1)U_{n-2}]$.

Solving the recursion

- $U_{n-1} - (n-1)\, U_{n-2} = \quad -[U_{n-2} - (n-2)\, U_{n-3}]$
- $U_{n-2} - (n-2)\, U_{n-3} = \quad -[U_{n-3} - (n-3)\, U_{n-4}]$
- … … … … … … …
- $U_3 - 3\, U_2 = -[\, U_2 - 2\, U_1]$, where $U_1 = 0$ & $U_2 = 1$.
- $U_n \quad - \quad n\, U_{n-1} \quad = \quad -[U_{n-1} - (n-1)\, U_{n-2}]$

  $= \quad (-1)^2\, [U_{n-2} \; - \; (n-2)\, U_{n-3}]$

  … …. ….. …..

  $= \quad (-1)^r\, [U_{n-r} \; - \; (n-r)\, U_{n-r-1}]$,

  $r = n-2, n-3, \ldots..2, 1.$

  $= (-1)^{n-2}[U_2 - 2U_1] = (-1)^n$

Solving the recursion

$$\frac{U_n}{\underline{|n}} - \frac{U_{n-1}}{\underline{|n-1}} = \frac{(-1)^n}{\underline{|n}} \quad \Rightarrow \quad \frac{U_2}{\underline{|2}} - \frac{U_1}{\underline{|1}} = +\frac{1}{\underline{|2}}$$

$$\frac{U_3}{\underline{|3}} - \frac{U_2}{\underline{|2}} = -\frac{1}{\underline{|3}}$$

$$\frac{U_4}{\underline{|4}} - \frac{U_3}{\underline{|3}} = +\frac{1}{\underline{|4}}$$

$$\cdots \quad \cdots \quad \cdots$$

$$\frac{U_n}{\underline{|n}} - \frac{U_{n-1}}{\underline{|n-1}} = \frac{(-1)^n}{\underline{|n}}$$

Adding, $$\boxed{\frac{U_n}{\underline{|n}} = \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \ldots + \frac{(-1)^n}{\underline{|n}}}$$

Approximation

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

For $n \geq 7$, this gives a fair approximation. For $n=7$, we have a correct value of 0.36786 and an approximation of 0.36788. Thus if we take the value of 0.3679 it is correct to 4 places of decimal.

So, for $n \geq 7$, we have $U_n = \text{floor}(n!e^{-1}) = \text{floor}(0.3679n!)$

An Example

- There are n distinct books and they are to be given to n students. They are given once the books, and then the books are returned. Then they are given the books for the second time. In how many cases will none of the students have the same book in both the distributions. Assume a large n.
- Answer is $(n!)^2e^{-1}$.