Principles of Counting

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Part-I

The Sum Rule

- Two tasks T_1 and T_2 are to be performed. If the task T_1 can be performed in m different ways and if the task T_2 can be performed in n different ways. The two tasks cannot be performed simultaneously, then one of the two tasks (T_1 or T_2) can be performed in m+n ways.
- This can be generalised to k tasks.

Examples

- Suppose there are 16 boys and 18 girls in a class. We wish to select one of these students (either a boy or a girl) as the class representative. The number of ways of selecting a student (boy or girl) is 16+18=34.
- Suppose a library has 12 books on Mathematics, 10 books on Physics, 16 books on Computer Sc and 11 books on Electronics. Suppose a student wishes to choose one of the books for study. He can do that in 12+10+16+11=49 ways.

Examples

- Suppose T_1 is the task of selecting a prime number less than 10. T_2 is say the task of selecting an even number less than 10. Thus $T_1=4$ and $T_2=4$.
- The task T of selecting a prime or even number less than 10 is 4+4-1=7 (why did we subtract 1?)

The Product Rule

- Suppose two tasks T_1 and T_2 are to be performed one after the other. If T_1 can be performed in n_1 different ways and for each of these ways T_2 can be performed in n_2 different ways, then both the tasks can be performed in $n_1 n_2$ different ways.

Examples

- A person has 3 shirts and 5 ties. Then he has 3 χ 5 different ways of choosing a tie and a shirt.
- Suppose we wish to construct a password of 4 symbols: first two alphabets and last two being numbers. The total number of passwords is:
  - 26x25x10x9 (wo repetitions)
  - 26^2x10^2 (with repetitions)

Examples

- Suppose a restaurant sells 6 South Indian dishes, 4 North Indian dishes, 3 hot and 4 cold beverages. If a student wants a breakfast which comprises of 1 South Indian dish and 1 hot beverage or 1 North Indian dish and 1 cold beverage, the total number of ways in which he chooses his breakfast is:
  - 6x3+4x4 ways. (apply both the sum and product rules).

Examples

- A telegraph can transmit two different signals: a dot and a slash. What length of those symbols is needed to encode 26 letters of the English alphabet and 10 digits.
- Number k length sequences is 2^k.
- So, the number of non-trivial sequences of length n or less is:

$$2+2^2+2^3+\ldots+2^n=2^{n+1}-2 \geq 36 \Rightarrow n \geq 5.$$

Examples

- Find the number of 3 digit even numbers with no repetition in digits.
- Let the number be xyz.
- z can be 0, 2, 4, 6, 8
- If z is 0, x cannot be 0, so it could be any of the 9 digits. y can be any of the 8 digits. So, there are 1x9x8=72 ways.
- If z is not 0, x cannot be either 0 or the value which z has taken. So, there are 8 choices for z. There are still 8 choices for y and hence there are 4x8x8=256 ways.
- Thus there are in total 72+256=328 ways (note that the choices are distinct and so we may apply the sum rule of counting).

Examples

- How many among the first 100,000 positive integers contain exactly one 3, one 4 and one 5 in their decimal representation.
- Answer is $5 \times 4 \times 3 \times 7 \times 7 = 2940$.

Examples

- Find the number of proper divisors of 441000.
- $441000 = 2^3 3^2 5^3 7^2$.
- Number of divisors are : $(3+1)(2+1)(3+1)(2+1)=144$.
- If we subtract 2 cases, for the divisors 1 and the number itself, we have 142 as the answer.

Permutations

- In some counting problems, order is important.
- Suppose we are given n distinct objects and we wish to arrange r of them in line. Since there are n ways of choosing the 1st object, then (n-1) ways of choosing the 2nd object and similarly the rth object may be chosen in (n-r+1) ways. Thus the total number of ways of choosing is:

$$n(n-1)\ldots(n-r+1)=(n!)/(n-r)!=P(n,r)$$

The number of different arrangements of n distinct objects is P(n,n)=n! It Is also called the permutation of n distinct objects.

Handling multisets

- The objects in a multiset may not be distinct (as in a set). Suppose we are required to find the arrangement of n objects of which n_1 are of one type, n_2 are of some other type and so on till n_k are of the k^{th} type=> $n_1+n_2+\ldots n_k=n$. Then the number of permutations of the n objects is:

$$P(n,(n_1,n_2,\ldots,n_k))=(n!)/(n_1!n_2!\ldots n_k!)$$

Example

- How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we wish to exceed 5,00,000?
- The number is of the form : $x_1x_2x_3x_4x_5x_6$
- x_1 can be either 5, 6, 7.
- When $x_1=5$, the remaining 5 digits have to be a permutation of digits from the multiset containing, 1(5),1(3),2(4),1(6) and 1(7). Thus the number of arrangements are: 5!/2!
- Answer is $(5!/2!)+(5!/2!^2)+(5!/2!^2)$.

Example

- In how many ways can n persons be seated at a round table if arrangements are considered the same when one can be obtained from the other by rotation?
- Let one of them be seated anywhere. The other (n-1) persons can be seated in (n-1)! ways. Thus the answer is (n-1)! ways.
- *Can you tell what is the difference between a linear and a circular arrangement?*

Examples

- Find the total number of positive integers that can be formed from the digits 1,2,3 and 4 if no digits are repeated in one integer.
- Note that the integer cannot contain more than 4 digits.
- Let it contain 1 digit. No of such numbers=4

Examples

- Let it contain 2 digits. No of such numbers=4x3=12.
- No of 3 digit integers=4x3x2=24
- No of 4 digit integers=4x3x2x1=24
- Hence the total number of numbers=4+12+24+24=64.

Examples on Divisibility

- If k is a positive integer, and n=2k, prove that $(n!)/2^k$ is a positive integers.
- Consider symbols: $x_1,x_1,x_2,x_2,\ldots,x_k,x_k$. The total number of symbols is 2k and there are k partitions of cardinality 2 each.
- So, the total number of arrangements in the multi-set: $(2k)!/(2!2!\ldots2!)=(2k)!/(2)^k$.
- Thus, $2^k \mid (2k)!$.

Another example

- Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$
- Set $N=n!$
- $(n-1)!=n!/n=N/n$.
- Thus we can consider a collection of N objects, which has $(n-1)!$ partitions of cardinality n each.
- The number of arrangements is $N!/(n!\ldots n!)=N!/(n!)^{(n-1)!}$. Thus, $(n!)^{(n-1)!}$ divides $N!$

Combinations

- Suppose, we are selecting (choosing) a set of r objects, from a set of $n \geq r$ objects without regard to order.
- The set of r objects being selected is traditionally called combination of r objects.

Relating to Permutations

- Let $C(n,r)$ be the combination of r distinct objects that can be selected from n different objects.
- The r objects chosen may be arranged in r! different ways.
- Thus, the total number of arrangements of the r objects chosen from the n distinct objects$=C(n,r)r!=P(n,r)$
- Thus $C(n,r)=P(n,r)/r!$
- Note that $C(n,r)=C(n,n-r)$

Examples

- How many committees of five with a given chairperson can be selected from 12 persons?
- Select the given chairperson.
- Select the remaining 4 members from the 11 persons (excluding the chairperson).
- Answer is C(11,4)

Examples

- Find the number of committees of 5 that can be selected from 7 men and 5 women if the committee is to consist of at least 1 man and 1 woman.
- Without restriction: C(12,5)
- Selections with 5 men: C(7,5)
- Selections with 5 women: C(5,5)
- Answer is C(12,5)-C(7,5)-C(5,5).

Examples

- Find the number of 5 digit positive integers such that in each of them every digit is greater than the digit to the right.
- A set of 5 distinct digits can be selected in C(10,5) ways.
- Of the digits selected there is only 1 arrangement, namely the descending order which is what we require.
- So, answer is 1xC(10,5).

Examples

- Find the number of ways of seating r out of n persons around a circular table and the others around another circular table.
- Answer is $C(n,r)(r-1)! \times (n-r-1)!$

Examples

- Find the number of arrangements of the letters in TALLAHASSEE which have no adjacent A's.
- The A's have to be placed in the dashes:

  —T—L—L—H—S—S—E—E—

- The number of possible arrangements of the remaining letters is $M=8!/(2!)^3$.
- The dashes can be filled in $N=C(9,3)$.
- Thus, the total number of arrangements=MN

Example

- How many arrangements are possible for 11 players, such that the batting order among A, B and C is A <<B<< C.
- Soln 1: Fix A in posn 1, B in posn 2. C can come in 9 places. Move B to 3, C can come in 8 places…so on till 1. Thus, there are 9x10/2 ways.
- Move A to posn 2, there are 8x9/2 ways.
- A can be in 9 positions, from 1 to 9.
- The other 8 players may be arranged in 8! ways.

- Thus there are in total: $8!\Sigma k(k+1)/2$, where k runs from 1 to 9.
- Soln2: There are 11 places. Select 3 positions for A, B and C. This can be done in C(11,3) ways. Of each selection there is only way which satisfies the ordering.
- The other places can be arranged in 8! ways.

Example

- Thus, there are in total C(11,3)x8! ways.
- Thus the result is 11!/3!
- Can you explain the answer in some other way?

Pascals Triangle

- Pascals formula:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Proof is simple: LHS is no of ways of selecting k distinct objects from n distinct objects.
- We can treat the problem in two mutually disjoint ways:
  - A (in which an object say x is always selected). This can be done in C(n-1,r-1) ways.
  - B (in which the object x is never selected). This can be done in C(n-1,r) ways.
- The proof follows from the sum rule.

Illustration of the proof

- n=5, k=3, S={x,a,b,c,d}
- A={(x,a,b),(x,a,c),(x,a,d),(x,b,c),(x,b,d),(x,c,d)}. Upon deleting x we obtain the number of possible ways of selecting 2 distinct objects from 4 distinct objects, C(4,2).
- B={(a,b,c),(a,b,d),(a,c,d),(b,c,d)}. The number of possible ways of selecting 3 distinct objects from 4 distinct objects, C(4,3).
- Thus result is C(4,2)+C(4,3) from the sum rule. This is equal to C(5,3).

Binomial Theorem

- Binomial Theorem for a positive integer n:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

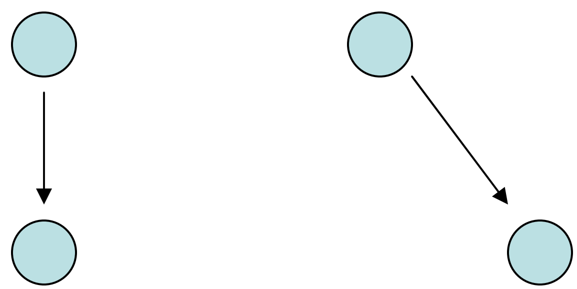
- The coefficients of $x^r y^{n-r}$ is called the binomial coefficient.

Computing Binomial Coefficients without Computing factorials

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

Another way of looking

- Define paths in the triangle with moves which are either vertically downward or which are diagonally to the right.



- Let Pth(n,k) be the number of paths from the entry C(0,0) to the entry C(n,k)

Another way of looking

- Pth(0,0)=1, Pth(n,0)=1, Pth(n,n)=1
- For k<n, Pth(n,k)=Pth(n-1,k-1)+Pth(n-1,k)
- Thus we have the same recurrence relation and the same initial conditions as C(n,r).
- Thus, Pth(n,r)=C(n,r)

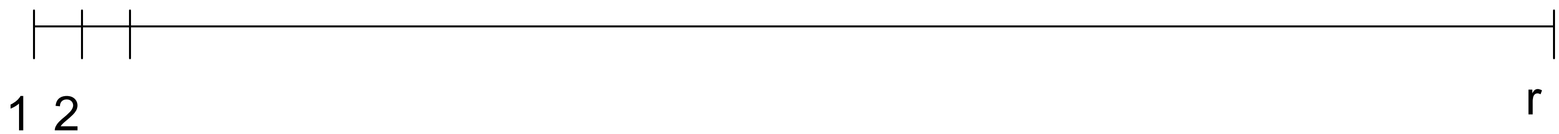
Multinomial Theorem

- For positive integers n and t, the coefficients of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \ldots x_t)^n$ is: $n!/(n_1!n_2!\ldots n_t!)$

Combinations of multisets

- Let S be a multiset with objects of k types, each with an infinite repetition number. Then the number of r combinations is equal to
  - C(r+k-1,k-1)=C(r+k-1,r)
- Proof Outline: The multiset is {∞(a1), ∞(a2),…, ∞(ak)}. Any r combination can be represented as the multiset {x1(a1),x2(a2),…xk(ak)} st. x1+x2+…+xk=r.
- We require integral solutions to the above equations.

Combinations of multisets



- We have to divide the line into k subparts.
- Add k-1 stickers labeled as ‘*’. Thus we have (r+k-1) distinct objects. Select (k-1) objects from them. Each selection will correspond to one such division.
- Thus, we have C(r+k-1,k-1) ways.

Examples

- A bakery boasts 8 varieties of doughnuts. If a box contains a dozen, how many boxes can you buy?
- Thus the bakery has 8 varieties, with a (possibly) large number of each.
- Select 12 dougnuts, with repetitions of a variety.
- Thus, we have $C(12+8-1,8-1)=C(12+8-1,12)$.