Principles of Counting

Debdeep Mukhopadhyay
IIT Madras
Part-I
The Sum Rule

- Two tasks $T_1$ and $T_2$ are to be performed. If the task $T_1$ can be performed in $m$ different ways and if the task $T_2$ can be performed in $n$ different ways. The two tasks cannot be performed simultaneously, then one of the two tasks ($T_1$ or $T_2$) can be performed in $m+n$ ways.
- This can be generalised to $k$ tasks.
Examples

- Suppose there are 16 boys and 18 girls in a class. We wish to select one of these students (either a boy or a girl) as the class representative. The number of ways of selecting a student (boy or girl) is $16+18=34$.

- Suppose a library has 12 books on Mathematics, 10 books on Physics, 16 books on Computer Sc and 11 books on Electronics. Suppose a student wishes to choose one of the books for study. He can do that in $12+10+16+11=49$ ways.
Examples

• Suppose $T_1$ is the task of selecting a prime number less than 10. $T_2$ is say the task of selecting an even number less than 10. Thus $T_1=4$ and $T_2=4$.

• The task $T$ of selecting a prime or even number less than 10 is $4+4-1=7$ (why did we subtract 1?)
The Product Rule

- Suppose two tasks $T_1$ and $T_2$ are to be performed one after the other. If $T_1$ can be performed in $n_1$ different ways and for each of these ways $T_2$ can be performed in $n_2$ different ways, then both the tasks can be performed in $n_1n_2$ different ways.
Examples

• A person has 3 shirts and 5 ties. Then he has $3 \times 5$ different ways of choosing a tie and a shirt.

• Suppose we wish to construct a password of 4 symbols: first two alphabets and last two being numbers. The total number of passwords is:
  – $26 \times 25 \times 10 \times 9$ (wo repetitions)
  – $26^2 \times 10^2$ (with repetitions)
Examples

- Suppose a restaurant sells 6 South Indian dishes, 4 North Indian dishes, 3 hot and 4 cold beverages. If a student wants a breakfast which comprises of 1 South Indian dish and 1 hot beverage or 1 North Indian dish and 1 cold beverage, the total number of ways in which he chooses his breakfast is:
  - 6x3+4x4 ways. (apply both the sum and product rules).
Examples

• A telegraph can transmit two different signals: a dot and a slash. What length of those symbols is needed to encode 26 letters of the English alphabet and 10 digits.

• Number \( k \) length sequences is \( 2^k \).

• So, the number of non-trivial sequences of length \( n \) or less is:

\[
2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2 \geq 36 \implies n \geq 5.
\]
Examples

- Find the number of 3 digit even numbers with no repetition in digits.
- Let the number be xyz.
- z can be 0, 2, 4, 6, 8
- If z is 0, x cannot be 0, so it could be any of the 9 digits. y can be any of the 8 digits. So, there are 1x9x8=72 ways.
- If z is not 0, x cannot be either 0 or the value which z has taken. So, there are 8 choices for z. There are still 8 choices for y and hence there are 4x8x8=256 ways.
- Thus there are in total 72+256=328 ways (note that the choices are distinct and so we may apply the sum rule of counting).
Examples

• How many among the first 100,000 positive integers contain exactly one 3, one 4 and one 5 in their decimal representation.
• Answer is $5 \times 4 \times 3 \times 7 \times 7 = 2940$. 
Examples

• Find the number of proper divisors of 441000.
• $441000 = 2^33^25^37^2$.
• Number of divisors are: $(3+1)(2+1)(3+1)(2+1) = 144$.
• If we subtract 2 cases, for the divisors 1 and the number itself, we have 142 as the answer.
Permutations

- In some counting problems, order is important.
- Suppose we are given n distinct objects and we wish to arrange r of them in line. Since there are n ways of choosing the 1st object, then (n-1) ways of choosing the 2nd object and similarly the rth object may be chosen in (n-r+1) ways. Thus the total number of ways of choosing is:
  \[ n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!} = P(n,r) \]

The number of different arrangements of n distinct objects is \( P(n,n) = n! \). It is also called the permutation of n distinct objects.
Handling multisets

• The objects in a multiset may not be distinct (as in a set). Suppose we are required to find the arrangement of $n$ objects of which $n_1$ are of one type, $n_2$ are of some other type and so on till $n_k$ are of the $k^{th}$ type => $n_1 + n_2 + \ldots + n_k = n$. Then the number of permutations of the $n$ objects is:

$$P(n,(n_1,n_2,\ldots,n_k)) = \frac{n!}{n_1!n_2!\ldots n_k!}$$
Example

• How many positive integers \( n \) can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we wish to exceed 5,00,000?
• The number is of the form: \( x_1x_2x_3x_4x_5x_6 \)
• \( x_1 \) can be either 5, 6, 7.
• When \( x_1=5 \), the remaining 5 digits have to be a permutation of digits from the multiset containing, 1(5),1(3),2(4),1(6) and 1(7). Thus the number of arrangements are: \( 5!/2! \)
• Answer is \( (5!/2!)+(5!/2!^2)+(5!/2!^2) \).
Example

• In how many ways can n persons be seated at a round table if arrangements are considered the same when one can be obtained from the other by rotation?

• Let one of them be seated anywhere. The other (n-1) persons can be seated in (n-1)! ways. Thus the answer is (n-1)! ways.

• Can you tell what is the difference between a linear and a circular arrangement?
Examples

• Find the total number of positive integers that can be formed from the digits 1,2,3 and 4 if no digits are repeated in one integer.

• Note that the integer cannot contain more than 4 digits.

• Let it contain 1 digit. No of such numbers=4
Examples

• Let it contain 2 digits. No of such numbers = $4 \times 3 = 12$.

• No of 3 digit integers = $4 \times 3 \times 2 = 24$

• No of 4 digit integers = $4 \times 3 \times 2 \times 1 = 24$

• Hence the total number of numbers = $4 + 12 + 24 + 24 = 64$. 
Examples on Divisibility

• If $k$ is a positive integer, and $n=2k$, prove that $(n!)/2^k$ is a positive integers.

• Consider symbols: $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$. The total number of symbols is $2k$ and there are $k$ partitions of cardinality 2 each.

• So, the total number of arrangements in the multi-set: $(2k)!/(2!2!\ldots2!)=(2k)!/(2)^k$.

• Thus, $2^k | (2k)!$. 
Another example

• Prove that \((n!)!\) is divisible by \((n!)^{(n-1)!}\)
• Set \(N=n!\)
• \((n-1)! = n!/n = N/n.\)
• Thus we can consider a collection of \(N\) objects, which has \((n-1)!\) partitions of cardinality \(n\) each.
• The number of arrangements is \(N!/(n!\ldots n!)=N!/(n!)^{(n-1)!}\). Thus, \((n!)^{(n-1)!}\) divides \(N!\)
Combinations

- Suppose, we are selecting (choosing) a set of $r$ objects, from a set of $n \geq r$ objects without regard to order.

- The set of $r$ objects being selected is traditionally called combination of $r$ objects.
Relating to Permutations

- Let $C(n,r)$ be the combination of $r$ distinct objects that can be selected from $n$ different objects.
- The $r$ objects chosen may be arranged in $r!$ different ways.
- Thus, the total number of arrangements of the $r$ objects chosen from the $n$ distinct objects = $C(n,r)r! = P(n,r)$
- Thus $C(n,r) = P(n,r)/r!$
- Note that $C(n,r) = C(n,n-r)$
Examples

• How many committees of five with a given chairperson can be selected from 12 persons?
• Select the given chairperson.
• Select the remaining 4 members from the 11 persons (excluding the chairperson).
• Answer is C(11,4)
Examples

• Find the number of committees of 5 that can be selected from 7 men and 5 women if the committee is to consist of at least 1 man and 1 woman.

• Without restriction: C(12,5)
• Selections with 5 men: C(7,5)
• Selections with 5 women: C(5,5)
• Answer is C(12,5)-C(7,5)-C(5,5).
Examples

• Find the number of 5 digit positive integers such that in each of them every digit is greater than the digit to the right.
• A set of 5 distinct digits can be selected in $C(10,5)$ ways.
• Of the digits selected there is only 1 arrangement, namely the descending order which is what we require.
• So, answer is $1 \times C(10,5)$. 
Examples

• Find the number of ways of seating \( r \) out of \( n \) persons around a circular table and the others around another circular table.

• Answer is \( \binom{n}{r}(r-1)! \times (n-r-1)! \)
Examples

• Find the number of arrangements of the letters in TALLAHASSEE which have no adjacent A’s.
• The A’s have to be placed in the dashes:
  —T—L—L—H—S—S—E—E—
• The number of possible arrangements of the remaining letters is $M=8!/(2!)^3$.
• The dashes can be filled in $N=C(9,3)$.
• Thus, the total number of arrangements $= MN$
Example

• How many arrangements are possible for 11 players, such that the batting order among A, B and C is A <<B<< C.
• Soln 1: Fix A in posn 1, B in posn 2. C can come in 9 places. Move B to 3, C can come in 8 places…so on till 1. Thus, there are 9x10/2 ways.
• Move A to posn 2, there are 8x9/2 ways.
• A can be in 9 positions, from 1 to 9.
• The other 8 players may be arranged in 8! ways.
• Thus there are in total: $8! \sum k(k+1)/2$, where $k$ runs from 1 to 9.

• Soln2: There are 11 places. Select 3 positions for A, B and C. This can be done in $C(11,3)$ ways. Of each selection there is only way which satisfies the ordering.

• The other places can be arranged in $8!$ ways.
Example

• Thus, there are in total $C(11,3) \times 8!$ ways.
• Thus the result is $11!/3!$
• Can you explain the answer in some other way?
Pascals Triangle

• Pascals formula:

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
\]

• Proof is simple: LHS is no of ways of selecting k distinct objects from n distinct objects.

• We can treat the problem in two mutually disjoint ways:
  – A (in which an object say x is always selected). This can be done in \( C(n-1,r-1) \) ways.
  – B (in which the object x is never selected). This can be done in \( C(n-1,r) \) ways.

• The proof follows from the sum rule.
Illustration of the proof

• n=5, k=3, S={x,a,b,c,d}
• A={(x,a,b),(x,a,c),(x,a,d),(x,b,c),(x,b,d),(x,c,d)}. Upon deleting x we obtain the number of possible ways of selecting 2 distinct objects from 4 distinct objects, C(4,2).
• B={a,b,c),(a,b,d),(a,c,d),(b,c,d)}. The number of possible ways of selecting 3 distinct objects from 4 distinct objects, C(4,3).
• Thus result is C(4,2)+C(4,3) from the sum rule. This is equal to C(5,3).
Binomial Theorem

• Binomial Theorem for a positive integer $n$:

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

• The coefficients of $x^r y^{n-r}$ is called the binomial coefficient.
Computing Binomial Coefficients without Computing factorials

<table>
<thead>
<tr>
<th>n/k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>4</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>
Another way of looking

- Define paths in the triangle with moves which are either vertically downward or which are diagonally to the right.

- Let $P_{th}(n,k)$ be the number of paths from the entry $C(0,0)$ to the entry $C(n,k)$
Another way of looking

- \( Pth(0,0)=1, \ Pth(n,0)=1, \ Pth(n,n)=1 \)
- For \( k<n \), \( Pth(n,k)=Pth(n-1,k-1)+Pth(n-1,k) \)
- Thus we have the same recurrence relation and the same initial conditions as \( C(n,r) \).
- Thus, \( Pth(n,r)=C(n,r) \)
Multinomial Theorem

• For positive integers $n$ and $t$, the coefficients of $x_1^{n_1} x_2^{n_2} x_3^{n_3} ... x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + ... x_t)^n$ is:
  \[ n!/(n_1!n_2!...n_t!) \]
Combinations of multisets

• Let $S$ be a multiset with objects of $k$ types, each with an infinite repetition number. Then the number of $r$ combinations is equal to
  \[- \binom{r+k-1}{k-1} = \binom{r+k-1}{r} \]
  
  • Proof Outline: The multiset is \{\infty(a_1), \infty(a_2), \ldots, \infty(a_k)\}. Any $r$ combination can be represented as the multiset \{x_1(a_1), x_2(a_2), \ldots x_k(a_k)\} s.t. \[x_1 + x_2 + \ldots + x_k = r.\]
  
  • We require integral solutions to the above equations.
Combinations of multisets

- We have to divide the line into $k$ subparts.
- Add $k-1$ stickers labeled as ‘*’. Thus we have $(r+k-1)$ distinct objects. Select $(k-1)$ objects from them. Each selection will correspond to one such division.
- Thus, we have $\binom{r+k-1}{k-1}$ ways.
Examples

• A bakery boasts 8 varieties of doughnuts. If a box contains a dozen, how many boxes can you buy?

• Thus the bakery has 8 varieties, with a (possibly) large number of each.

• Select 12 doughnuts, with repetitions of a variety.

• Thus, we have $C(12+8-1,8-1) = C(12+8-1,12)$. 