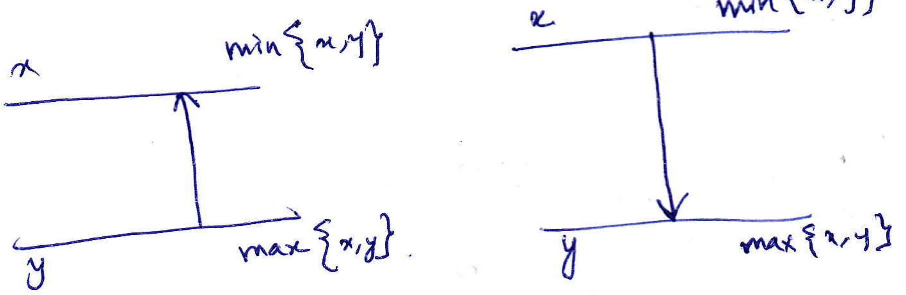
## Sorting Networks.

- Special purpose hardware for sorting.
- Comparator network model for handling comparision problems well suited.

We shall study <u>bitonic sorting network</u>.

What is a Comparator network?

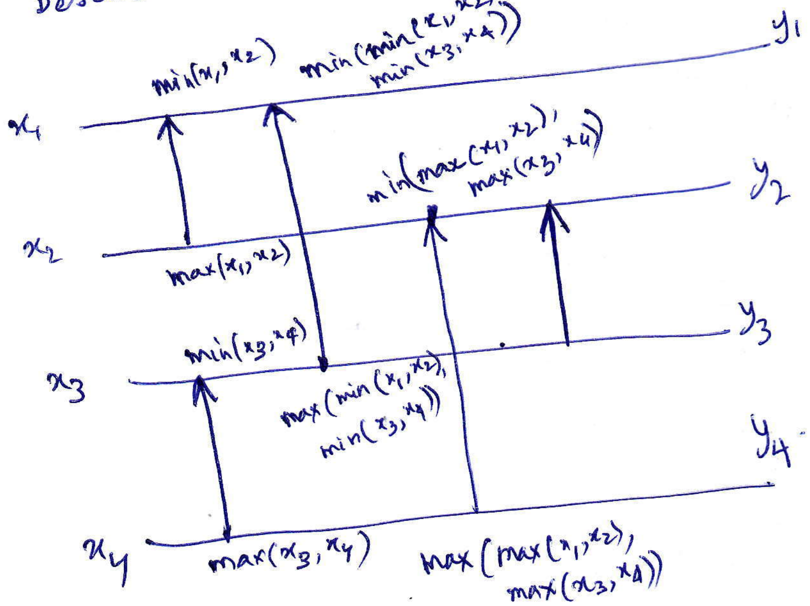
Made up of comparators, where a comparator is a module whose two i/ps are $x$ and $y$ and whose two ordered outputs are $\min\{x,y\}$ and $\max\{x,y\}$.




Size of comparator network : Number of comparators

Depth : Maximum number of comparators encountered when going from an i/p to o/p.

Describe a sorting network for $x_1, x_2, x_3, x_4$.



<u>Goal</u>

- Design comparator networks that sort in small depths and that have small size.
- Given such a network, we can easily derive a parallel algorithm 1) whose run time is same as depth.

  2) Total number of operations: size of the network.

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<u>Bitonic Sequences.</u>

Let $X = (x_0, x_1 \ldots\ldots x_{n-1})$ be a sequence of elements drawn from a linearly ordered set.

$\boxed{\text{Assume } n = 2^k,\ k \in I}$.

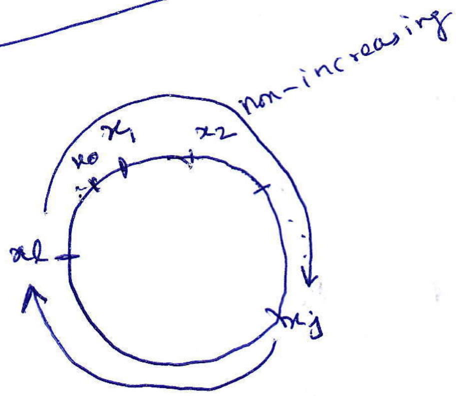
<u>Def$^n$</u> Seq. $X$ is called bitonic if $\exists$ some +ve integer $j < n$ we have

$x_{j \bmod n} \leq x_{(j+1)\% n} \ldots \leq x_{l \bmod n}$

&

$x_{(l+1)\% n} \geq \ldots\ldots \geq x_{(j+n-1)\% n}$

for some $l$.




<u>Ex</u> $X = (\underset{x_0}{-5}, \underset{x_1}{-9}, \underset{x_2}{-10}, \underset{x_3}{-5}, \underset{x_4}{2}, \underset{x_5}{7}, \underset{x_6}{35}, \underset{x_7}{37})$ non-decreasing

$j = 2$
$l = 6$
(in the Def$^n$).

$-10 < -5 < 7 < 35$.

$37 > -5 > -9$

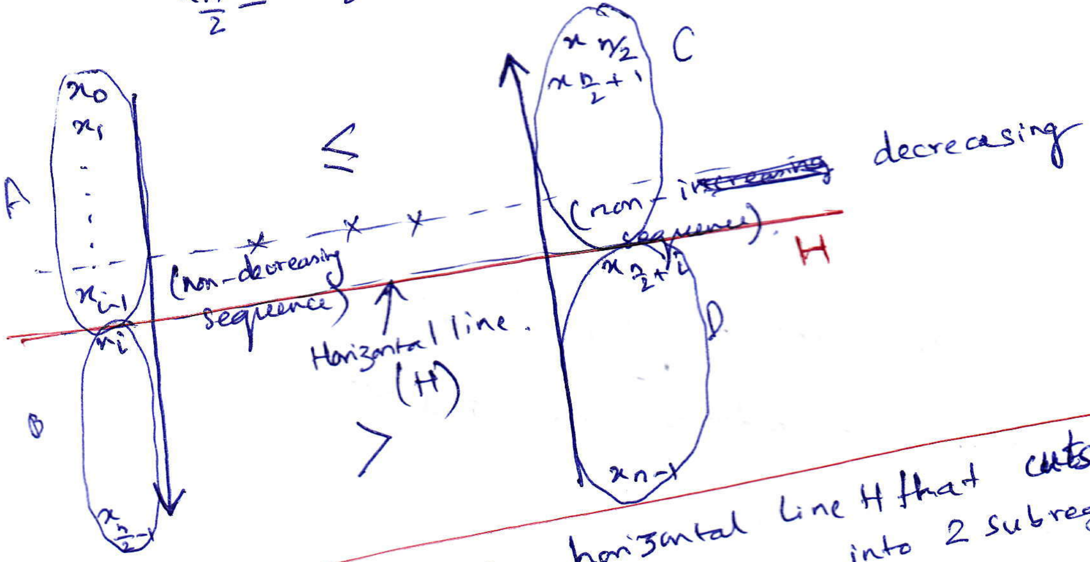
# Unique Cross Over property of Bitonic Sequences.

Let us start with a special bitonic sequence

$$X = (x_0, x_1, x_2, \ldots x_{n-1})$$

s.t. $x_0 \le x_1 \le \ldots \le x_{\frac{n}{2}-1}$ &

$x_{\frac{n}{2}} \ge x_{\frac{n}{2}+1} \ge \ldots \ge x_{n-1}$.




There exists a unique horizontal line H that cuts the region between two vertical arrows into 2 subregions: one labeled ≤ and other labeled >

1) an arbitrary pair of elements in the same subregion, one from left column, and other from right column satisfies relation indicated in the subregion.

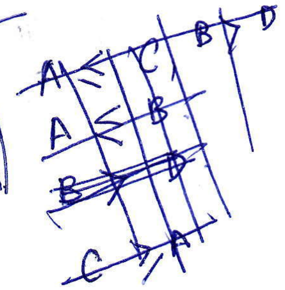
∴ Eg, $x_1 \le x_{n/2+1}$ & $x_i > x_{n/2+i}$

2) every element in the left column of the subregion ≤ is less than or equal to every element in the subregion > [Eg $x_1 \le x_i$].

in the right column.

Property 2 also holds

Prove, $x_{n/2+i} \le x_{\frac{n}{2}+1}$



Justification.

$$
\begin{array}{ccc}
x_0 & & x_{\frac{n}{2}} \\
x_1 & & x_{\frac{n}{2}+1} \\
x_j & \leq & \\
x_{i-1} & & x_{\frac{n}{2}+i-1} \\
\hline
x_i & & x_{\frac{n}{2}+i} \\
x_K & > & \\
x_{n/2-1} & & x_{n-1}
\end{array}
$$

Suppose, $i$ is the smallest index, st. $\boxed{x_i > x_{\frac{n}{2}+i}}$

Consider the horizontal cut between $x_{i-1}$ and $x_i$

(and between $x_{\frac{n}{2}+i-1}$ and $x_{\frac{n}{2}+i}$)

$x_j \leq x_{i-1} \leq x_{\frac{n}{2}+i-1} \leq x_{\frac{n}{2}+i-2} \leq \dots \leq x_{n/2}$.

$\therefore$ For $0 \leq j \leq i-1$,

$\therefore$ $x_j$ is less than or equal to each element of the right hand side in the subregion $\leq$.

Likewise, for $i \leq K \leq \frac{n}{2}-1$,

$x_K \geq x_i > x_{\frac{n}{2}+i} \geq x_{\frac{n}{2}+i+1} \geq \dots \, x_{n-1}$

$\therefore$ $x_K$ is greater than each element of the right hand side in the subregion $>$

Also, $x_j$ is less than or equal to each element of the left ~~edge~~ hand side in the subregion $\leq$.

Sorting of Bitonic Sequences.

Given a bitonic sequence $X = (x_0\, x_1 \dots x_{n-1})$.

we define:

$L(X) = \min\{x_0, x_{\frac{n}{2}}\}, \min\{x_1, x_{\frac{n}{2}+1}\}, \dots, \min\{x_{\frac{n}{2}-1}, x_{n-1}\}$.

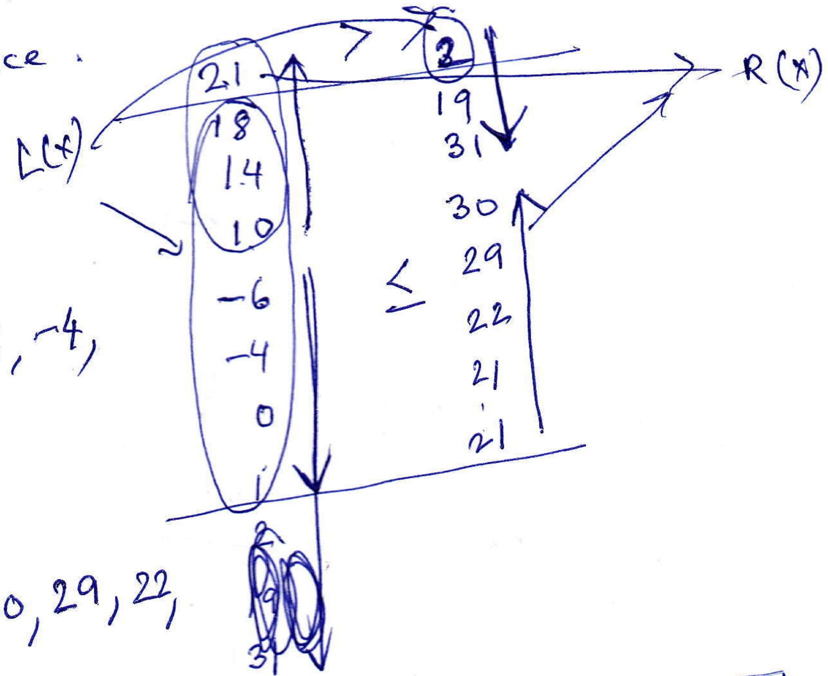
$R(X) = \max\{x_0, x_{n/2}\}, \max\{x_1, x_{n/2+1}\}, \dots \max\{x_{n/2-1}, x_{n-1}\}$.

Ex $X = (21, 18, 14, 10, -6, -4, 0, 1, 2, 19, 31, 30, 29, 22, 21, 21)$

be a bitonic sequence.

$L(X) = ?$

$R(X) = ?$




$L(X) = (2, 18, 14, 10, -6, -4, 0, 1)$

$R(X) = (21, 19, 31, 30, 29, 22, 21, 21)$.

> Note: Each of L(X) and R(X) consists of a segment in the subregion ≤ and the complement subregion >

Let, X be a bitonic sequence, ~~Let~~ Then, both L(X) and R(X) are bitonic, and each element of L(X) is smaller than each element of R(X)

Proof

$$\begin{array}{cc} x_0 & x_{\frac{n}{2}} \\ x_1 & x_{\frac{n}{2}+1} \\ \vdots & \le \\ x_i & x_{i+n/2} \\ \hline x_{i+1} & x_{i+1+n/2} \\ \vdots & > \\ x_{\frac{n}{2}-1} & x_{n-1} \end{array}$$

Given the properties of Horizontal Cuts

$L(X) = (x_0, x_1, \ldots, x_i, x_{i+1+n/2}, \ldots, x_{n-1})$

$R(X) = (x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, \ldots, x_{i+n/2}, x_{i+1}, \ldots, x_{n/2-1})$

Since, L(X) and R(X) are subsequences of the bitonic sequence X, clearly L(X) and R(X) are both bitonic.

## Algorithm

Input: A bitonic sequence $X = (a_0 \ldots x_{n-1})$ such that $n = 2^k$ for some integer k.

Output: The sequence X in sorted order.

begin

1. for $0 \leq i \leq n/2 - 1$ parallel do

   set $l_i = \min(x_i, x_{n/2+i})$

   $r_i = \max(x_i, x_{n/2+i})$

2. Apply the algorithm recursively to each of the sequences $L(X) = (l_0\ l_1 \ldots l_{n/2-1})$ and $R(X) = (r_0, r_1, \ldots r_{n/2-1})$.

3. Output the sorted sequence L(X) followed by the sorted sequence R(X)

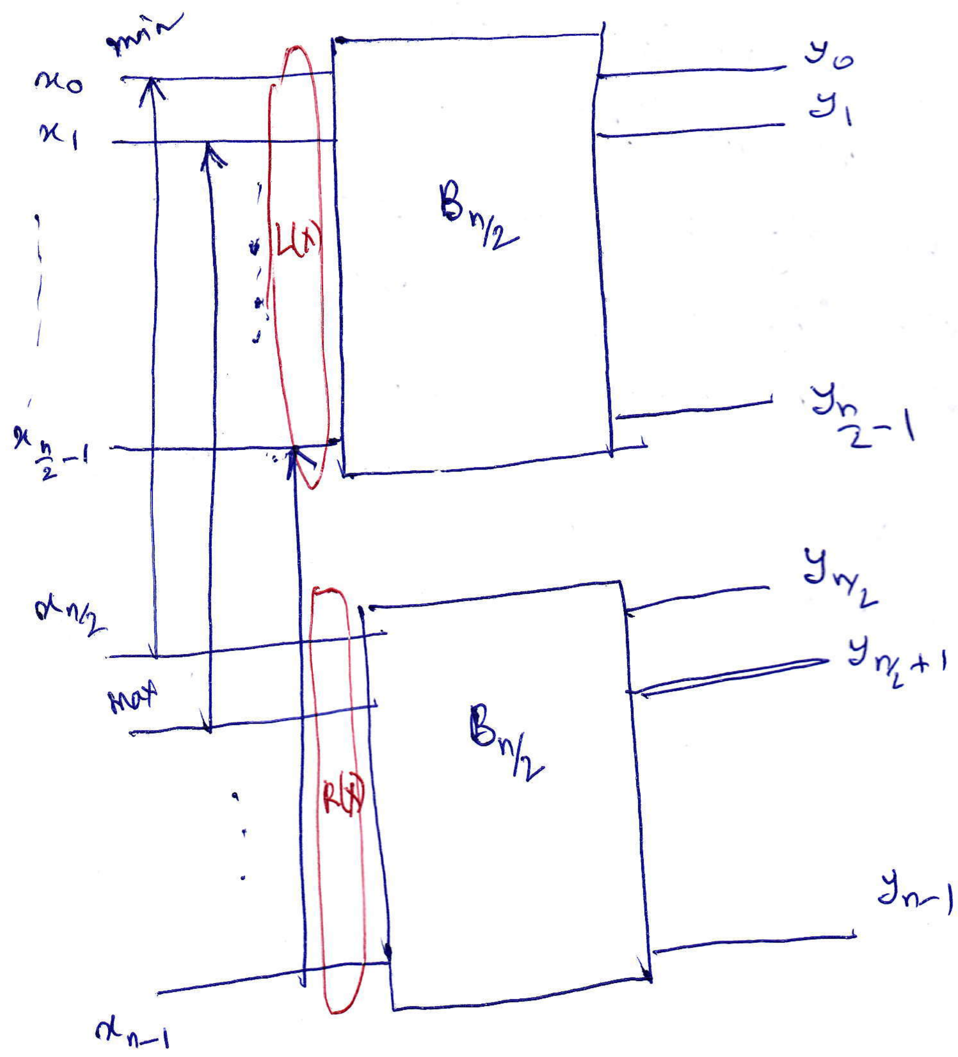
End.

## Bitonic Sorting Network [Knuth].

$B(n)$: to sort bitonic sequence of length $n = 2^k$, for some integer $k$.

If $n = 2$, then $B(2)$ consists of just one comparator

$n > 2$, Consider the following network



Depth: $D(n) = 1 + D(n/2)$ $\Big\}$ $D(n) = \log n$.

$D(2) = 1$

Comparator Cost: $C(n) = n/2 + 2C(n/2)$ $\Big\}$ $C(n) = \frac{n \log n}{2}$.

$C(2) = 1$

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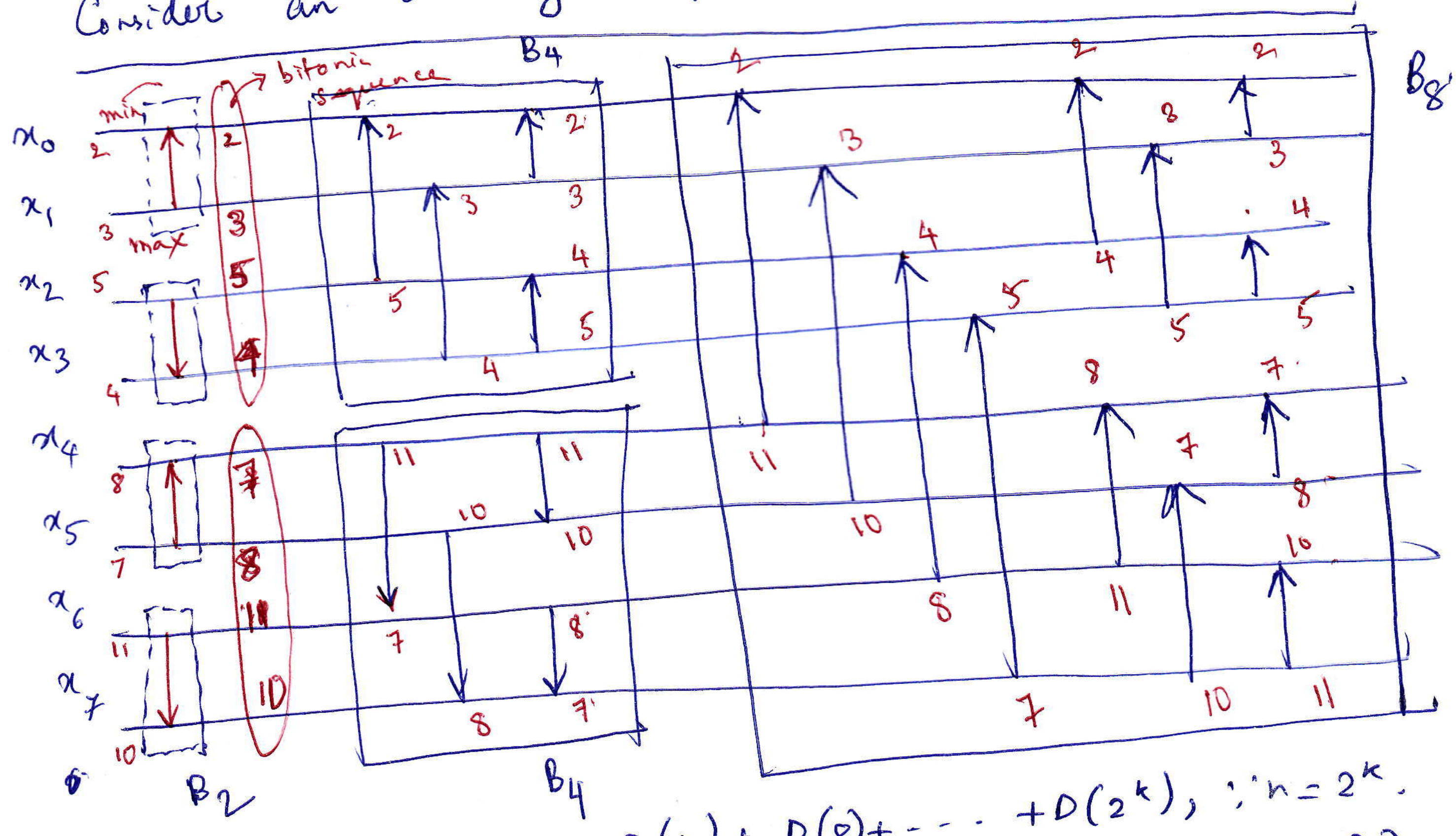
NB: Bitonic Sorting Network sorts bitonic sequences, and does not necessarily sort arbitrary sequences.

How can you sort arbitrary sequences?

Consider an arbitrary sequence $X = (x_0\, x_1 \dots x_{n-1})$




$$\text{Depth} = D(2) + D(4) + D(8) + \dots + D(2^k), \ \because n = 2^k.$$
$$= 1 + 2 + 3 + \dots + \log n = O(\log^2 n)$$

$$\text{Number of Comparators} = \frac{n}{2} C(2) + \frac{n}{4} C(4) + \frac{n}{8} C(8) + \dots + \frac{n}{2^k} D(2^k)$$
$$= \sum \frac{n}{2^i} C(2^i) = O(n \log^2 n).$$