$$x_i = \begin{cases} b_0 & i = 0 \\ (x_{i-1} \otimes a_i) \oplus b_i & 0 < i < n \end{cases}$$ first order recurrences.

1. $\oplus$ is associative.
2. $\otimes$ is semi-associative. (i.e $\exists$ a binary associative operator $\odot$ such that $(a \otimes b) \otimes c = a \otimes (b \odot c)$)
3. $\otimes$ distributes over $\oplus$ (i.e $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$).

The operator $\odot$ is called companion operator of $\otimes$.

Q Can we reduce it to

$$x_i = \begin{cases} a_i & i = 0 \\ x_{i-1} \oplus a_i & 0 < i < n \end{cases}$$

Consider the set of pairs $c_i = [a_i, b_i]$.

Define a new binary operator $\odot$:

$$c_i \bullet c_j \equiv [c_{i,a} \odot c_{j,a},\ (c_{i,b} \otimes c_{j,a}) \oplus c_{j,b}]$$

where $c_{i,a}$, $c_{i,b}$ are first and second elements of c_i respectively.

Prove: $(c_i \bullet c_j) \bullet c_k = c_i \bullet (c_j \bullet c_k)$.

$$\text{LHS} = [c_{i,a} \odot c_{j,a},\ (c_{i,b} \otimes c_{j,a}) \oplus c_{j,b}] \bullet c_k$$

$$= [(c_{i,a} \odot c_{j,a}) \odot c_{k,a},\ (((c_{i,b} \otimes c_{j,a}) \oplus c_{j,b}) \otimes c_{k,a}) \oplus c_{k,b}]$$

$$= [c_{i,a} \odot (c_{j,a} \odot c_{k,a}),\ ((c_{i,b} \otimes c_{j,a}) \otimes c_{k,a} \oplus ((c_{j,b} \otimes c_{k,a}) \oplus c_{k,b})]$$

$$= [c_{i,a} \odot (c_{j,a} \odot c_{k,a}),\ (c_{i,b} \otimes (c_{j,a} \odot c_{k,a})) \oplus ((c_{j,b} \otimes c_{k,a}) \oplus c_{k,b})]$$

[semi-associativity of $\otimes$]

$$= c_i \bullet [c_{j,a} \odot c_{k,a},\ (c_{j,b} \otimes c_{k,a}) \oplus c_{k,b}]$$

$$= c_i \bullet (c_j \bullet c_k)$$

QED.

Now, define the ordered set $s_i = [y_i, x_i]$ where y_i obey the recurrence:

$$y_i = \begin{cases} a_0 & i = 0 \\ y_{i-1} \odot a_i & 0 < i < n. \end{cases}$$

Then, $S_0 = [y_0, x_0]$
$= [a_0, b_0]$
$= c_0.$

$c_i = [a_i, b_i]$
$c_{i,a} = a_i$
$c_{i,b} = b_i$

$$\begin{aligned} S_i &= [y_i, x_i] \\ &= [y_{i-1} \odot a_i, \ (x_{i-1} \otimes a_i) \oplus b_i] \\ &= [y_{i-1} \odot c_{i,a}, \ (x_{i-1} \otimes c_{i,a}) \oplus c_{i,b}] \\ &= [y_{i-1}, x_{i-1}] \cdot c_i \\ &= s_{i-1} \cdot c_i \end{aligned}$$

Since $\cdot$ is associative we reduce it to a form which can be solved by the scan algorithm.

EREW PRAM

$$T = \left(\frac{n}{p} + \lg p\right).$$

Higher order Recurrences.

$$x_i = \begin{cases} b_i & 0 \le i \le m \\ (x_{i-1} \otimes a_{i,1}) \oplus \cdots \oplus (x_{i-m} \otimes a_{i,m}) \oplus b_i & m \le i \le n. \end{cases}$$

$\oplus$ is associative.

$\otimes$ is semi-associative

$\otimes$ distributes over $\oplus$.

Define $\quad S_i = [x_i \;\cdots\; x_{i-m+1}]$

$$\therefore\; S_i = \underbrace{[x_{i-1} \;\cdots\; x_{i-m}]}_{1\times m} \otimes_v \underbrace{\begin{bmatrix} a_{i,1} & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ \vdots & 0 & \cdots & & 1 \\ a_{i,m} & 0 & \cdots & & 0 \end{bmatrix}}_{m\times m} \oplus_v [b_i \; 0 \cdots 0].$$

$$= (S_{i-1} \otimes_v A_i) \oplus_v B_i.$$

$\otimes_v$ is a vector-matrix multiply

$\oplus_v$ is a vector addition.

We can use matrix-matrix multiply as the companion operator of $\otimes_v$

$$[x_{i-1} \; \cdots \; x_{i-m}] \otimes_\vartheta \begin{bmatrix} a_{i,1} & \cdots & \\ & & \\ a_{i,m}\,0 & \cdots & 0 \end{bmatrix}$$

$$\boxed{(a \otimes_\vartheta A) \otimes_\vartheta B = a \otimes_\vartheta (A \otimes_m B)}$$

$$\left([a_1 \cdots a_m] \otimes_\vartheta \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \\ A_{m,1} & & A_{m,m} \end{bmatrix} \right) \otimes_\vartheta \begin{bmatrix} B_{1,1} & \cdots & B_{1,m} \\ & & \\ B_{m,1} & & B_{m,m} \end{bmatrix}$$

$$= [a_1 A_{1,1} \oplus \cdots \oplus a_m A_{m,1} \mid \cdots \mid a_1 A_{1,m} \oplus \cdots \oplus a_m A_{m,m}] \otimes_\vartheta \begin{bmatrix} B_{1,1} & \cdots & B_{1,m} \\ & & \\ B_{m,1} & \cdots & B_{m,m} \end{bmatrix}.$$

$$= [(a_1 A_{1,1} \oplus \cdots \oplus a_m A_{m,1}) B_{1,1} \oplus \cdots \oplus (a_1 A_{1,m} \oplus \cdots \oplus a_m A_{m,m}) B_{m,1} \mid \cdots \mid (a_1 A_{1,1} \oplus \cdots \oplus a_m A_{m,1}) B_{1,m} \oplus \cdots \oplus (a_1 A_{1,m} \oplus \cdots \oplus a_m A_{m,m}) B_{m,m}]$$

$$= [a_1 (A_{1,1} B_{1,1} \oplus \cdots \oplus A_{1,m} B_{m,1}) \oplus a_2 \cdots \oplus a_m (A_{m,1} B_{1,1} \oplus \cdots \oplus A_{m,m} B_{m,1}) \mid \cdots \mid a_1 (A_{1,1} B_{1,m} \oplus \cdots \oplus A_{1,m} B_{m,m}) \oplus \cdots \oplus a_m (A_{m,1} B_{1,m} \oplus \cdots \oplus A_{m,m} B_{m,m})]$$

$$= [a_1 \cdots a_m] \left(\begin{bmatrix} A \end{bmatrix} \otimes_m \begin{bmatrix} B \end{bmatrix} \right).$$

X_s is semi-associative with logical or as a

Companion operator

$$x_i = (x_{i-1}\ X_s\ f_i) \oplus a_i$$

$$(a\ X_s\ f_1)\ X_s\ f_2 = a\ X_s\ (f_1 \vee f_2)$$

f_1	f_2	LHS	RHS
0	0	a	a
0	1	0	0
1	0	0	0
1	1	0	0

$$x_i = \begin{cases} a_0 & i = 0 \\ \begin{cases} a_i & f_i = 1 \\ (x_{i-1} \oplus a_i) & f_i = 0 \end{cases} & 0 < i < n \end{cases}$$

$$x_i = \begin{cases} a_0, & i = 0 \\ (x_{i-1}\ X_s\ f_i) \oplus a_i & 0 < i < n \end{cases}$$

where X_s is defined as:

$$x\ X_s\ f = \begin{cases} I_{\oplus} & f = 1 \\ x & f = 0 \end{cases}$$