

# Partial Orderings

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## Definition

- A relation  $R$  on a set  $S$  is called a **partial ordering** if it is reflexive, antisymmetric and transitive.
- A set  $S$  together with a partial ordering  $R$  is called a **partially ordered set**, or poset, and is denoted by  $(S,R)$ .
- Let  $X = \{1,2,3,4,5,6\}$  and  $P = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (6,1), (6,4), (1,4), (6,5), (3,4), (6,2)\}$ . Then  $P$  is partial order on  $X$ , and  $(X,P)$  is a poset.

## Example

- Show that “greater than or equal” relation is a partial ordering on the set of integers.
  - $a \geq a$  for every integer  $a$  (reflexive)
  - $a \geq b, b \geq a$ , then  $a = b$  (anti-symmetric)
  - $a \geq b, b \geq c$ , then  $a \geq c$  (transitive)
- Thus  $\geq$  is a partial ordering on the set of integers
- $(\mathbb{Z}, \geq)$  is a poset.

## Examples

- Similarly, the division symbol ‘|’ is a partial ordering on the set of positive integers.
- The inclusion relation  $\subseteq$  is a partial ordering on the set of  $P(S)$
- In a poset, the notation  $a \preceq b$ , indicates  $aRb$ .
- The notation,  $a \prec b$  means that  $a \preceq b$ , but not  $a = b$ .

## Comparable and Incomparable

- The elements  $a$  and  $b$  of a poset  $(S, \preceq)$  are called comparable, if either  $a \preceq b$  or  $b \preceq a$ . When  $a$  and  $b$  are elements of  $S$  such that neither  $a \preceq b$  or  $b \preceq a$ , they are called incomparable.
- In the Poset  $(\mathbb{Z}^+, |)$ , are the integers 3 and 9 comparable? Yes, as  $3|9 \Rightarrow 3 \preceq 9$ .
- But 5 and 7 are incomparable.

## Totally Ordered Sets

- If  $(S, \preceq)$  is a poset and every two elements of  $S$  are comparable,  $S$  is called a totally ordered set or linearly ordered set.
- It is also called a chain.
- The Poset  $(\mathbb{Z}, \leq)$  is a chain.
- The Poset  $(\mathbb{Z}^+, |)$  is not a chain.

## Well Ordered Set

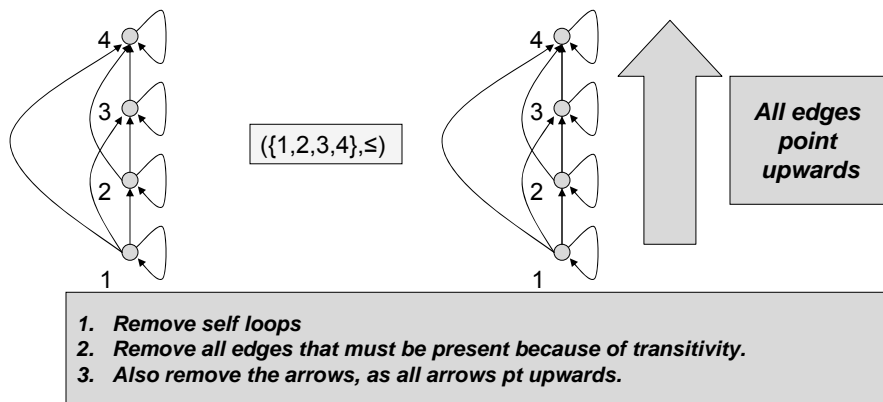
- $(S, \preceq)$  is a well ordered set if it is a poset such that  $\preceq$  is a total ordering and such that every non-empty subset of  $S$  has a least element.
- Set of ordered pairs of positive integers,  $Z^+ \times Z^+$ , with  $(a_1, a_2) \preceq (b_1, b_2)$  if  $a_1 \leq b_1$  or  $a_1 = b_1$  and  $a_2 \leq b_2$ .
- The set  $Z$  with the usual  $\leq$  ordering, is not well ordered.
- *Finite sets which are Totally ordered sets are well ordered.*

## Lexicographic Order

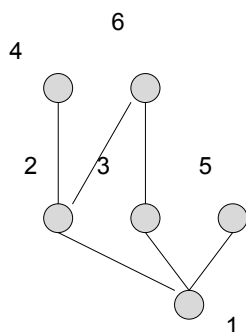
- Define an ordering on  $A_1 \times A_2$  by specifying that one pair is less than the other, if
  - The first entry of the first pair is less than the first entry of the second pair, or
  - If the first entries are equal, but the second entry of the first pair is less than the second entry of the second pair.
  - To make it partial ordering add equality to the ordering.

# Hasse Diagram

- We can represent a Poset by a directed graph.



# Hasse Diagram ( $\{1,2,3,4,5,6\}, |$ )



- Hasse Diagram for the relation  $R$  represents the smallest relation  $R'$  such that  $R=(R')^*$

## Quasi Order

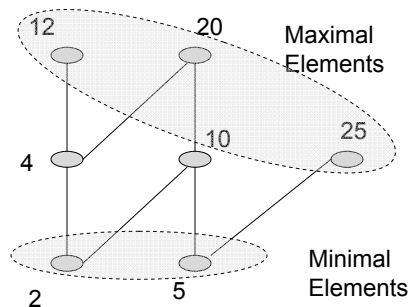
- Let  $R$  be a binary relation on  $A$ .  $R$  is a quasi order if  $R$  is transitive and irreflexive. The only distinction between a quasi order and a partial order is the equality relation.
- $R$  is always anti-symmetric. Why?
- Example:
  - The relation  $<$  on the set of real numbers.
  - The relation “is a prerequisite” is a quasi order on any set of college courses.
  - PERT chart represents a quasi order on the collection of tasks to be performed.  $xRy$  means that task  $y$  cannot be started until task  $x$  is finished.

## Maximal and Minimal Elements

- Maximal: An element  $a$  of a poset  $(S, \preceq)$  is maximal if there is no element  $b$  in  $S$ , st  $a \preceq b$ .
- Similarly, we also have a minimal element in the poset.
- They are respectively, the “top” and the “bottom” elements in the diagram.

## Example

- Which elements of the poset  $(\{2,4,5,10,12,20,25\},|)$  are maximal and which are minimal?



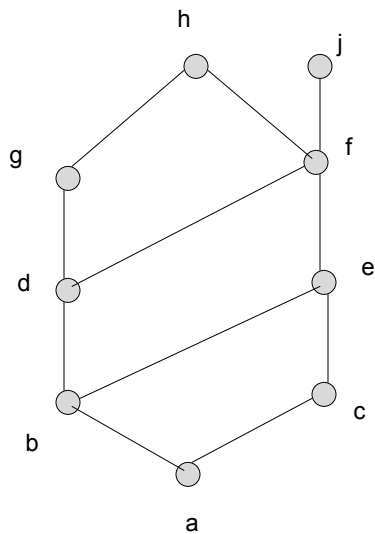
## More terms

- **Greatest element:** Sometimes there is an element in a poset that is the greatest than every other elements.
- **Least element:** Sometimes there is an element which is less than all other elements in the poset.
- *The greatest and least element, when they exist are unique.*

## Bounds

- Sometimes it is possible to find an element, that is greater than all the elements in a subset  $A$  of  $(S, \preceq)$ . Then it is called the *upper bound* of  $A$ .
- Similarly, we have a *lower bound* of  $A$ .
- Least Upper Bound  $\text{lub}(A)$ : Least among the upper bounds. If it exists, it is unique.
- Greatest lower Bound  $\text{glb}(A)$ : Greatest among the lower bounds.

## Example



- $\text{UB}(\{a,b,c\}) = \{e,f,j,h\}$
- $\text{LB}(\{a,b,c\}) = a$
- $\text{UB}(\{j,h\}) = \{ \}$
- $\text{LB}(\{j,h\}) = \{a,b,c,d,e,f\}$
- $\text{UB}(\{a,c,d,f\}) = \{h,f,j\}$
- $\text{LB}(\{a,c,d,f\}) = \{a\}$
- $\text{glb}(\{b,d,g\}) = \max(\{a,b\}) = b$
- $\text{lub}(\{b,d,g\}) = \min(\{g,h\}) = g$



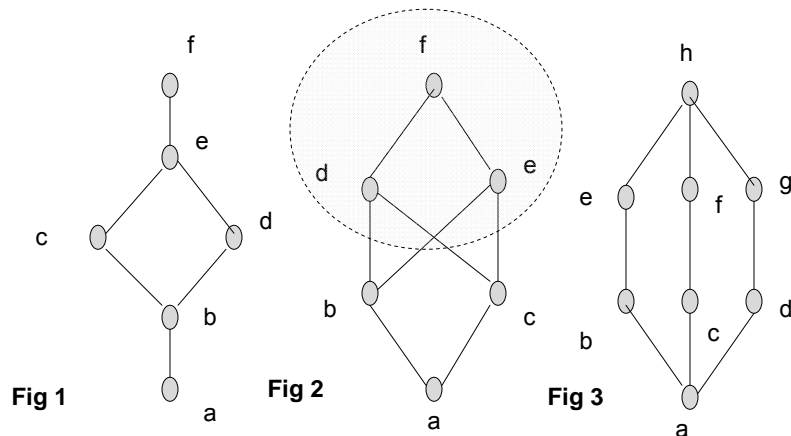
## More Examples

- Find the glb and lub of the sets  $\{3,9,12\}$  and  $\{1,2,4,5,10\}$  if they exist in the poset  $(\mathbb{Z}^+, |)$ .
- $\text{glb}=3$
- $\text{lub}=36$ .

## Lattices

- **Lattices:** A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound.
- They are very useful as models of information flow and Boolean algebra.

## Which one are lattices?



- Fig 1 and 3 are lattices. Fig 2 is not because,  $\{b,c\}$  has no lub
- However, it has a  $\text{glb}=\{a\}$

## Topological Sorting

- A total ordering is said to be **compatible** with the partial ordering  $R$  if  $a \preceq b$  whenever  $aRb$ .
- $aRb \Rightarrow a \preceq b$  (Partial  $\Rightarrow$  Total)
- Constructing a compatible total ordering from a partial ordering is called **topological sorting**.

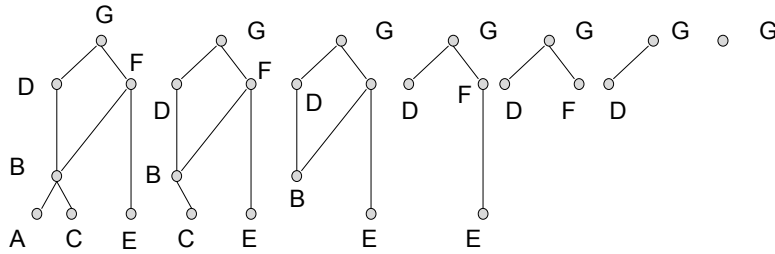
## Theorem







- Every finite non-empty poset  $(S, \preceq)$  has a minimal element.
- Proof is left as an exercise. It follows from the fact that the set is finite and so our search for an minimal element should terminate at one point.

## Algorithm

```
procedure topological sort(S:finite poset)
k=1
while S≠∅
begin
  ak=a minimal element of S{such an element
  exists by lemma 1}
  S=S-{ak}
  k=k+1
end{a1,a2,...,an is a compatible total ordering of S}
```

## Example of a Topological Sort



- A  C  B  E  F  D  G

Minimal Element Chosen