Construction of Pseudo-random Functions

Debdeep Mukhopadhyay
IIT Kharagpur

Background

- We have seen how to make Pseudo-random generators from one way functions.
- We shall proceed to make Pseudo-random functions from generators.
- Let G be a PSRG with expansion factor $l(n)=2n$ (i,e G is length doubling)
- Define, $G(s)=(G_0(s),G_1(s))$, where $|s|=|G_0(s)|=|G_1(s)|=n$.

- Use G to make keyed function F
  - uses an n bit key
  - takes one bit as input
  - outputs another n bits
- For a key k, define,
  - $F_k(0)=G_0(k)$
  - $F_k(1)=G_1(k)$
- We claim that this is a pseudorandom function! Why?

Simple Reason

- This follows from the fact that G is a pseudorandom generator.
- A random function mapping one bit to n bits is defined by a table of two n-bit values, each of which is random.
- Here we have defined a keyed function, where each n-bit value is pseudorandom

  (as the key is randomly chosen)
- Thus F_k cannot be distinguished from a random function by a PPT algorithm.

Extend to two bit input

- $F_k(00)=G_0(G_0(k))$
- $F_k(01)=G_1(G_0(k))$
- $F_k(10)=G_0(G_1(k))$
- $F_k(11)=G_1(G_1(k))$
  - in order to show that F_k is pseudorandom, thus we have to reason that the four strings, $G_0(G_0(k))$, $G_1(G_0(k))$, $G_0(G_1(k))$, $G_1(G_1(k))$ are pseudorandom.

Hybrid Construction

- $G_0(G_0(k)) \rightarrow G_0(k_0) \rightarrow r_1$
- $G_1(G_0(k)) \rightarrow G_1(k_0) \rightarrow r_2$
- $G_0(G_1(k)) \rightarrow G_0(k_1) \rightarrow r_3$
- $G_1(G_1(k)) \rightarrow G_1(k_1) \rightarrow r_4$

- Here k_0, k_1, r_1, r_2, r_3 and r_4 are randomly chosen n bit strings.

Hybrid Construction

- $G_0(G_0(k)) \rightarrow G_0(k_0) \rightarrow r_1$
- $G_1(G_0(k)) \rightarrow G_1(k_0) \rightarrow r_2$
- $G_0(G_1(k)) \rightarrow G_0(k_1) \rightarrow r_3$
- $G_1(G_1(k)) \rightarrow G_1(k_1) \rightarrow r_4$

If you can distinguish between these strings then you can distinguish either between $G(k_0)$ and (r_1,r_2), or $G(k_1)$ and (r_1, r_2)

If you can distinguish between these strings then you can distinguish between $G(k)=(G_0(k),G_1(k))$ and (k_0, k_1)

as both of these contradicts the pseudo-randomness of G

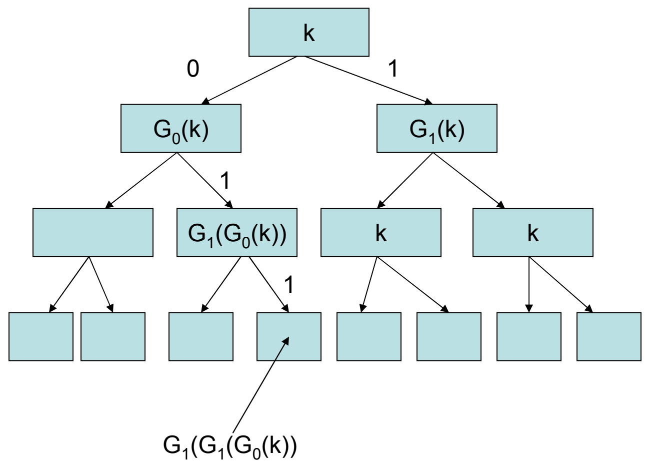
Combining, these facts we have F_k as pseudorandom.

More generalization

Define: $F_k : \{0,1\}^n \rightarrow \{0,1\}^n$

$$F_k(x_1x_2...x_n) = G_{x_n}(G_{x_{n-1}}(...G_{x_1}(k)))$$

Pictographically



Explanation

- The construction can be viewed as a full binary tree of depth n.
- The value at the root is the key k.
- The value of a left child of a node with value k' is $G_0(k')$
- The value of a right child of a node with value k' is $G_1(k')$
- The value of $F_k(x)$ is thus obtained by traversing the tree according to x
  - if $x_i=0$ traverse left
  - else traverse right
- The entire tree is exponential in n.
  - however to compute the function the entire tree need not be stored. we just need to compute the values on the path and arrive at a leaf.

Theorem

- If G is a pseudorandom generator with expansion factor *l(n)=2n*, then the above construction is a pseudorandom function.

Proof

- Let D be a PPT algorithm which is given oracle access to a function that is either a random function that maps n bits to n bits, or the function F_k for a randomly chosen k.

Proof

- Consider the distribution of trees, obtained by varying the leaf randomly.
- Each leaf of the binary tree of depth n, is thus a sequence of n bits.
- use H_n^0 to denote the distribution.
  - note this is the distribution may be thought of being on the functions F_k.

Proof

- Likewise, define H_n^i, for $0 \le i \le n$ as follows:
  - values for node i is chosen at random.
  - values for nodes $j \ge i+1$ are chosen as per the function definition. That is see the value of its parent. If the value is k':
    - value is $G_0(k')$ if is left child
    - value is $G_1(k')$ if is right child
  - note that from the point of view of the function, the values of the nodes at levels 0 through i-1 are irrelevant. This is because they do not decide the value of the leaves.

What is H_n^n?

- It is a true random function mapping n bits to n bits.
  - this is because all the leaf values are randomly chosen.
- So, the distinguisher D is able to distinguish between the distribution H_n^0 (the actual construction) and H_n^n (the random function)

Construct D' (distinguisher against G)

- Assume that D (distinguisher against the PRF F_k) makes t(n) queries to the function.
- The output is of length 2n.t(n)
- Thus D' has 2n.t(n) bits of either truly random bits or output generated by t(n) invocations of the function G(s), for a randomly chosen s.

Strategy of D'

- D' answers queries of D as follows:
  - D asks queries of the form $x_1x_2\ldots x_n$
  - D' chooses a i randomly, and goes to node i of the initially empty binary tree.
  - It computes the values of the nodes at level i+1 with its sample of length 2n as follows:
    - labels the left node with left part of the sample
    - labels the right node with right part of the sample

Observations

- If D' receives a truly random string of length 2nt(n), then it answers D exactly according to H_n^{i+1}. Why?
- If D' receives a pseudorandom input, then it answers D exactly according to H_n^i.
- Thus, if for some i, D distinguishes H_n^i and H_n^{i+1} with a probability of ε(n)/n, then with the same probability D' also distinguishes t(n) invocations of G(s) from a truly random string of length 2n.t(n) with probability ε(n)/n.
- If ε(n) is negligible, we violate the assumption that G is a PRG.

Reading

- How to Construct Random Functions?
  - O. Goldreich, Goldwasser, Micali, JACM 1986

One way functions

- If one way functions then pseudo random generators exist.
- If pseudorandom generators exist, so does pseudorandom functions.
- One way functions are hence necessary.
- Are one way functions sufficient also?

Theorem

- Pseudorandom generators exist only if one-way functions exist or

If there are pseudorandom generators, then there exists one-way functions.

Let G be a pseudo-random generator with expansion factor of length 2n. We show G is itself one-way. We shall show that the inability to invert G can be used to distinguish the output of G from random.

Proof

Let A be a PPT algorithm, and then define:

$$\varepsilon(\mathrm{n})=\Pr[\mathrm{Invert}_{\mathrm{A,G}}(n)=1]$$

Define D a PPT as follows:

Input : $\mathrm{w} \in \{0,1\}^{2\mathrm{n}}$

1. Run A on w. x=A(w)
2. If w=G(x), then return 1, else 0.

Computing the success probability of D.

If w is random, what is the probability that D returns 1?

Note that there are at most 2^n elements in the range of G. If w falls outside the range, then A cannot invert and so D answers 0. Hence, $\Pr_{w \leftarrow \{0,1\}^{2n}}[D(w)=1] \leq 2^{-n}$

Conclusion

If w=G(s) for a uniformly chosen s, then by definition A computes a correct inverse, with probability exactly ε(n). This is the same probability with which D returns 1.

$\therefore |\Pr_{w \leftarrow \{0,1\}^{2n}}[D(w)=1] - \Pr_{s \leftarrow \{0,1\}^{n}}[D(G(s))=1]| \geq \varepsilon(n) - 2^{-n}$.

Hence, if ε(n) is negligible, then D also have a significant success probability.

Question

- Does secured private key encryption imply the existence of one-way functions?
  - not straightforward
  - there may be construction techniques which do not depend on the above primitives.
- We show that it really does, assuming the weakest form of security notions of the encryption scheme.

Theorem

- If there exists a private key encryption that has indistinguishable encryptions in the presence of an eavesdropper, the one-way functions exist!
  - note for a perfect cipher, where the key length is same or more than the message length, such an assumption need not hold.
  - so we are considering practical ciphers, where the key length is less than the message length.

Proof

Define Π=(Gen,Enc,Dec) be a private key encryption scheme that has indistuishable encryptions in the presence of an adversary.
Define f: $f(k,m,r)=(\mathrm{Enc}_k(m,r), m)$
Here k, m and r are respectively of n, 2n and l(n) bits. That is the encryption uses at most l(n) bits of randomness. We claim that this function is one-way.

Proof

Consider a PPT algorithm A, which inverts the function, f with a probability of ε(n).

$\therefore \varepsilon(n)=\Pr[\mathit{Invert}_{A,f}(n)=1]$. Assume ε(n) is non-negligible.

Now define a PPT algorithm A', which runs an experiment $\mathrm{Priv}^{\mathrm{CPA}}_{\mathrm{A},\Pi'}(n)$.

Now define a PPT algorithm A', which runs an experiment $\text{Priv}^{\text{CPA}}_{\text{A},\Pi'}(\text{n})$.

1. A' chooses random $\text{m}_0,\text{m}_1 \leftarrow \{0,1\}^{2\text{n}}$ and output the two messages. It receives a challenge c, which is the encryption of m_b, where b is randomly chosen.

2. A' has to say whether b=0 or 1. A' runs $\text{A}(\text{c},\text{m}_0)$ to obtain (k',m',r'). If $\text{f}(\text{k}',\text{m}',\text{r}')=(\text{c},\text{m}_0)$, then A' outputs 0. Else it outputs a random bit.

If c has been generated by encrypting m_0 $[i,e\ b=0]$ and A is able to invert, then we see that A' gives correct answer. Otherwise, if A is unable to invert, A' has a probability of 1/2 being correct.

$$\therefore \Pr[\text{Priv}^{CPA}_{\Pi,\text{A}'}(n)=1\,|\,b=0]=\Pr[invert_A\,|\,b=0]+\frac{1}{2}(1-\Pr[invert_A\,|\,b=0])$$

$$=\varepsilon(\text{n})+\frac{1}{2}(1-\varepsilon(\text{n}))=\frac{1}{2}(1+\varepsilon(\text{n}))$$

Proof

If c has been generated by encrypting m_1 (i,e b=1) by a key say k, what is the probability that A' returns 1?
Note that c must be the ciphertext of the message m_0 for some other value of the key, say k'. So, when (c,m_0) is being given to A, the probability that c is actually the ciphertext of a randomly chosen m_0 is atmost $2^n.2^{-2n} = 2^{-n}$. Then A inverts and obtains (k',m_0,r''), and if $f(k',m_0,r'')=(c,m_0)$, then it returns 0. Now this wrong, as b=1. Otherwise, invert does not take place and there is 1/2 probability of A' to return the correct bit.

Conclusion

$$\therefore \Pr[\mathrm{Priv}_{\Pi,A'}^{CPA}(n)=1|b=0]=\frac{1}{2}(1-\Pr[\mathrm{Invert}_A|b=1]) \geq \frac{1}{2}(1-2^{-n})$$

$$\text{Combining, } \Pr[\mathrm{Priv}_{\Pi,A'}^{CPA}(n)=1] \geq \frac{1}{2}.\frac{1}{2}(1+\varepsilon(n))+\frac{1}{2}.\frac{1}{2}(1-2^{-n})$$

$$=\frac{1}{2}+\frac{\varepsilon(n)}{4}-\frac{1}{2^{n+2}}$$

Thus the indistinguishability of the encryption scheme under the assumption of an eavesdropper is violated. Thus ε(n) must be negligible.