# The Discrete Logarithm Problem

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# The DLP Problem

- Consider $\alpha \varepsilon G$, having order n.
  - $<\alpha>=\{\alpha^i:0\leq i\leq n-1\}$ is a cyclic sub-group of G having order n.

**Problem 6.1:** **Discrete Logarithm**

**Instance:** A multiplicative group $(G, \cdot)$, an element $\alpha \in G$ having order $n$, and an element $\beta \in \langle \alpha \rangle$.

**Question:** Find the unique integer $a$, $0 \leq a \leq n-1$, such that

$$\alpha^a = \beta.$$

We will denote this integer $a$ by $\log_\alpha \beta$; it is called the *discrete logarithm* of $\beta$.

# Cryptographic Utility of DLP

- For suitable choices of the parameters, finding Discrete Logarithm seems to be difficult.
- However, the inverse operation of exponentiation is efficiently computable by the square and multiply algorithm.
  - Exponentiation is a candidate one-way function.

# The ElGamal Cryptosystem

**Cryptosystem 6.1:** *ElGamal Public-key Cryptosystem in* $\mathbb{Z}_p{}^*$

Let $p$ be a prime such that the **Discrete Logarithm** problem in $(\mathbb{Z}_p{}^*, \cdot)$ is infeasible, and let $\alpha \in \mathbb{Z}_p{}^*$ be a primitive element. Let $\mathcal{P} = \mathbb{Z}_p{}^*$, $\mathcal{C} = \mathbb{Z}_p{}^* \times \mathbb{Z}_p{}^*$, and define

$$\mathcal{K} = \{(p, \alpha, a, \beta) : \beta \equiv \alpha^a \pmod{p}\}.$$

The values $p$, $\alpha$ and $\beta$ are the public key, and $a$ is the private key.

For $K = (p, \alpha, a, \beta)$, and for a (secret) random number $k \in \mathbb{Z}_{p-1}$, define

$$e_K(x, k) = (y_1, y_2),$$

where

$$y_1 = \alpha^k \bmod p$$

and

$$y_2 = x\beta^k \bmod p.$$

For $y_1, y_2 \in \mathbb{Z}_p{}^*$, define

$$d_K(y_1, y_2) = y_2(y_1{}^a)^{-1} \bmod p.$$

# Working of the algorithm

- Plaintext x is masked by multiplying it by $\beta^k$, yielding $y_2$.
- The value $\alpha^k$ is also transmitted as a part of the ciphertext.
- Bob who has the secret ‘a’ can compute $\beta^k$ by raising $\alpha^k$ to ‘a’.
- Then he obtains x by dividing $y_2$ with $\beta^k$
- Note that for each plaintext, there are p-1 possible ciphertexts.

# Example

- $p=2579$, $\alpha=2$ (primitive element of $Z_p^*$)
- $a=765$ (secret value)
- $\beta=2^{765} \bmod 2579=949$.
- Suppose, Alice wishes to send $x=1299$ to Bob. She randomly chooses $k=853$.
  - $y_1=2^{853} \bmod 2579 = 435$
  - $y_2=1299(949^{853}) \bmod 2579=2396$
- Alice sends $y=(435,2396)$
- Bob computes $x=2396(435^{765})^{-1} \bmod 2579=1299$.

# Algorithms for the DLP Problem

- If $\alpha^i$ was monotonically non decreasing with i, we could have done a binary search to find i.
  - but the problem with modular exponentiation is that there is no ordering of the powers.
  - Thus one have to do an exhaustive search in the worst case.
    - Thus it can be solved in O(n) time and O(1) space.
  - However pre-computation helps.

# Time Memory Trade Off

- Suppose we store all possible values of $\alpha^i \pmod p$ as ordered pairs $(i, \alpha^i \bmod p)$ and sort the elements wrt the second parameter. Now search for the given challenge by employing binary search.
- Complexity: Pre-computation O(n), Memory O(n), Time to sort: O(nlog n) [using a good sorting algorithm], Time to search O(log n)
- Often we neglect the log n terms in these algorithms, as n is much larger than log n
  - thus Time to search O(1) and Pre-computation or Memory both are O(n)

# Non-trivial Algorithms

- **Shank's Algorithm**
- Pollard Rho Discrete Log Algorithm
- Index Calculus Method

# Shanks Algorithm

**Algorithm 6.1:** SHANKS$(G, n, \alpha, \beta)$

1. $m \leftarrow \lceil \sqrt{n} \rceil$
2. **for** $j \leftarrow 0$ **to** $m - 1$
   **do** compute $\alpha^{mj}$
3. Sort the $m$ ordered pairs $(j, \alpha^{mj})$ with respect to their second coordinates, obtaining a list $L_1$
4. **for** $i \leftarrow 0$ **to** $m - 1$
   **do** compute $\beta\alpha^{-i}$
5. Sort the $m$ ordered pairs $(i, \beta\alpha^{-i})$ with respect to their second coordinates, obtaining a list $L_2$
6. Find a pair $(j, y) \in L_1$ and a pair $(i, y) \in L_2$ (i.e., find two pairs having identical second coordinates)
7. $\log_\alpha \beta \leftarrow (mj + i) \bmod n$

# Explanation

- $\alpha^{mj}=y=\beta\alpha^{-i} \Rightarrow \alpha^{mj+i}=\beta$.
- If $\beta \varepsilon \langle\alpha\rangle$, $\log_\alpha \beta=(mj+i) \bmod n$, where both $0 \le i,j \le m-1$.
  - The search is successful as we can ensure that $\log_\alpha \beta \le m(m-1)+(m-1)=n-1$, as desired.
  - Complexity: O(m)

# Example

- Compute, $\log_3 525$ in $Z_{809}{}^*$. Note 809 is prime and 3 is a primitive element of $Z_{809}{}^*$.
- Order=n=808, $\beta$=525, m=$\sqrt{808}$=29
- $\alpha^{29}$ mod 809=99

# Tables

L1:

| | | | | |
|---|---|---|---|---|
| $(0,1)$ | $(1,99)$ | $(2,93)$ | $(3,308)$ | $(4,559)$ |
| $(5,329)$ | $(6,211)$ | $(7,664)$ | $(8,207)$ | $(9,268)$ |
| $(10,644)$ | $(11,654)$ | $(12,26)$ | $(13,147)$ | $(14,800)$ |
| $(15,727)$ | $(16,781)$ | $(17,464)$ | $(18,632)$ | $(19,275)$ |
| $(20,528)$ | $(21,496)$ | $(22,564)$ | $(23,15)$ | $(24,676)$ |
| $(25,586)$ | $(26,575)$ | $(27,295)$ | $(28,81)$ | |

L2:

| | | | | |
|---|---|---|---|---|
| $(0,525)$ | $(1,175)$ | $(2,328)$ | $(3,379)$ | $(4,396)$ |
| $(5,132)$ | $(6,44)$ | $(7,554)$ | $(8,724)$ | $(9,511)$ |
| $(10,440)$ | $(11,686)$ | $(12,768)$ | $(13,256)$ | $(14,355)$ |
| $(15,388)$ | $(16,399)$ | $(17,133)$ | $(18,314)$ | $(19,644)$ |
| $(20,754)$ | $(21,521)$ | $(22,713)$ | $(23,777)$ | $(24,259)$ |
| $(25,356)$ | $(26,658)$ | $(27,489)$ | $(28,163)$ | |

- **Match:**

  (10,644) and (19,644)

  $\log_3 525 = (29\text{x}10+19) \bmod 808 = 309$.

# The Diffie Hellman Problem

**Problem 6.3: Computational Diffie-Hellman**

**Instance:** A multiplicative group $(G, \cdot)$, an element $\alpha \in G$ having order $n$, and two elements $\beta, \gamma \in \langle\alpha\rangle$.

**Question:** Find $\delta \in \langle\alpha\rangle$ such that $\log_\alpha \delta \equiv \log_\alpha \beta \times \log_\alpha \gamma \pmod{n}$. (Equivalently, given $\alpha^b$ and $\alpha^c$, find $\alpha^{bc}$.)

**Problem 6.4: Decision Diffie-Hellman**

**Instance:** A multiplicative group $(G, \cdot)$, an element $\alpha \in G$ having order $n$, and three elements $\beta, \gamma, \delta \in \langle\alpha\rangle$.

**Question:** Is it the case that $\log_\alpha \delta \equiv \log_\alpha \beta \times \log_\alpha \gamma \pmod{n}$? (Equivalently, given $\alpha^b$, $\alpha^c$ and $\alpha^d$, determine if $d \equiv bc \pmod{n}$.)

- DDH $<<_P$ CDH $<<_P$ DLP
- Thus DDH hardness is the strongest assumption.

## Application: The DH Key Agreement Scheme

- **Public:** g and p
- **Secret:** Alice's exponent a, Bob's exponent b

$g^a$ mod p →

← $g^b$ mod p

Alice, a Bob, b

- Alice computes $(g^b)^a = g^{ba} = g^{ab}$ mod p
- Bob computes $(g^a)^b = g^{ab}$ mod p
- Could use $K = g^{ab}$ mod p as symmetric key

## Application: The DH Key Agreement Scheme

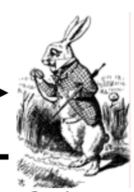
- Subject to man-in-the-middle (MiM) attack

$g^a$ mod p →

← $g^t$ mod p

$g^t$ mod p →

← $g^b$ mod p



Alice, a Trudy, t Bob, b

- Trudy shares secret $g^{at}$ mod p with Alice
- Trudy shares secret $g^{bt}$ mod p with Bob
- Alice and Bob don't know Trudy exists!

## Designing Cryptographic Protocols

- The Man in the Middle Attack on the DH key agreement scheme shows that although the primitives are strong, the protocol can be weak.
- Thus, the next question is how to design strong protocols from strong primitives.
- We will not discuss in depth, but have a brief overview as our last topic this semester…

## Possible Preventions

- How to prevent MiM attack?
  – Encrypt DH exchange with symmetric key
  – Encrypt DH exchange with public key
  – Sign DH values with private key
  – May be other methods also exist