Epidemic models

1. Epidemic models
   - SI model
   - SIS model
   - SIR model

Utility:

Models information diffusion, rumor spread, disease propagation etc.

- Mathematical epidimiology
- W. O. Kermack and A. G. McKendrick, 1927
- Deterministic compartamental model (population classes) $\{S, I, T\}$
- $S(t)$ - succeptable, number of individuals not yet infected with the disease at time t
- $I(t)$ - infected, number of individuals who have been infected with the disease and are capable of spreading the disease.
- $R(t)$ - recoverd, number of individuals who have been infected and then recovered from the disease, can't be infected again or to transmit the infection to others.
- Fully-mixing model
- Closed population (no birth, death, migration)
- Models: SI, SIS, SIR, SIRS,..

SI Model

- $S(t)$ -susceptible , $I(t)$ - infected

$$S \longrightarrow I$$

$$S(t) + I(t) = N$$

- β - infection/contact rate, Probability of infecting a susceptible node
- Infection equation:

$$I(t+\delta t) = I(t) + \beta \frac{S(t)}{N} I(t) \delta t$$

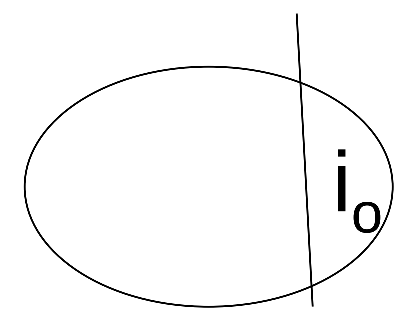
$$\frac{dI(t)}{dt} = \beta \frac{S(t)}{N} I(t)$$

SI Model

- Fractions: $i(t) = I(t)/N$, $s(t) = S(t)/N$
- Equations

$$\frac{di(t)}{dt} = \beta s(t) i(t)$$
$$\frac{ds(t)}{dt} = -\beta s(t) i(t)$$

$$s(t) + i(t) = 1$$



- Differential equation, $i(t=0) = i_0$

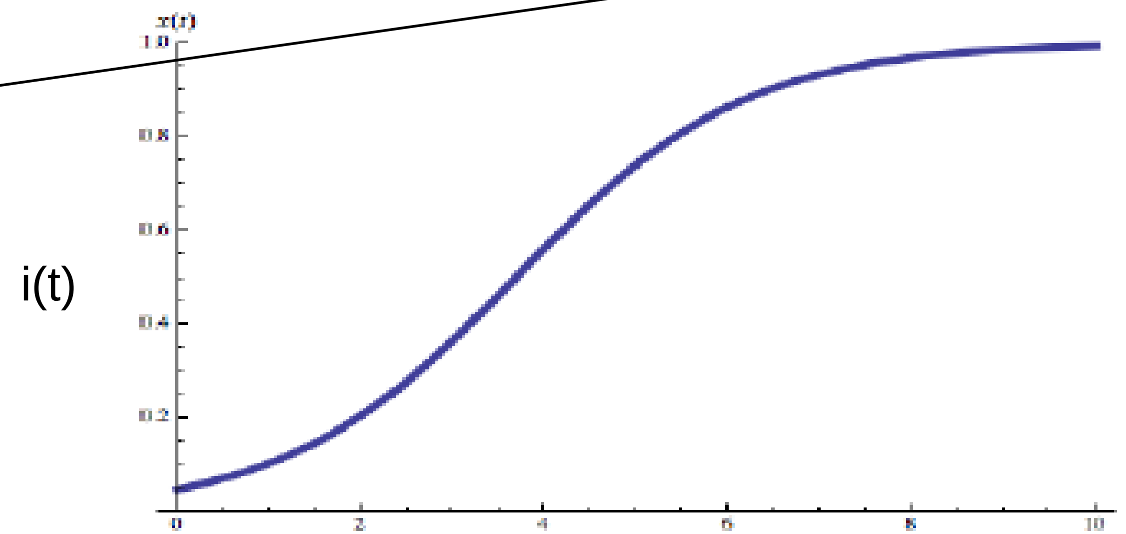
$$\frac{di(t)}{dt} = \beta(1 - i(t))i(t)$$

SI model

- Solution:

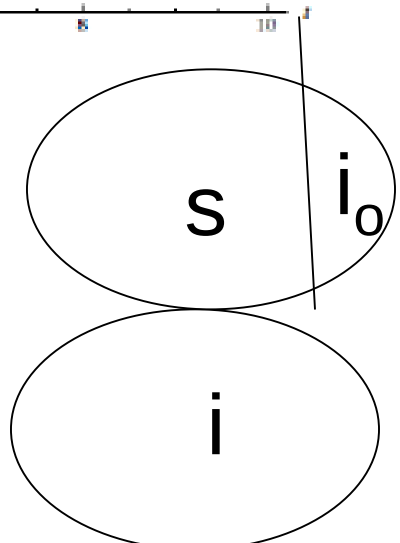
$$i(t) = \frac{i_0}{i_0 + (1 - i_0)e^{-\beta t}}$$

1. i_o=0??
2. beta>0
   t->infy
   i(t)->1



- Limit $t \to \infty$

$$i(t) \to 1$$
$$s(t) \to 0$$



in image $i_0 = 0.05, \quad \beta = 0.8$

SIS model

- $S(t)$ -susceptable , $I(t)$ - infected,

$$S \longrightarrow I \longrightarrow S$$

$$S(t) + I(t) = N$$

- β - infection rate (on contact), γ - recovery rate
- Infection equations:

$$\begin{aligned} \frac{ds}{dt} &= -\beta si + \gamma i \\ \frac{di}{dt} &= \beta si - \gamma i \end{aligned}$$

$$s + i = 1$$

- Differential equation, $i(t = 0) = i_0$

$$\frac{di}{dt} = (\beta - \gamma - \underset{\beta}{i})i$$

SIS model

- Solution

$$i(t) = (1 - \frac{\gamma}{\beta}) \frac{C}{C + e^{-(\beta-\gamma)t}}$$

where

$$C = \frac{\beta i_0}{\beta - \gamma - \beta i_0}$$

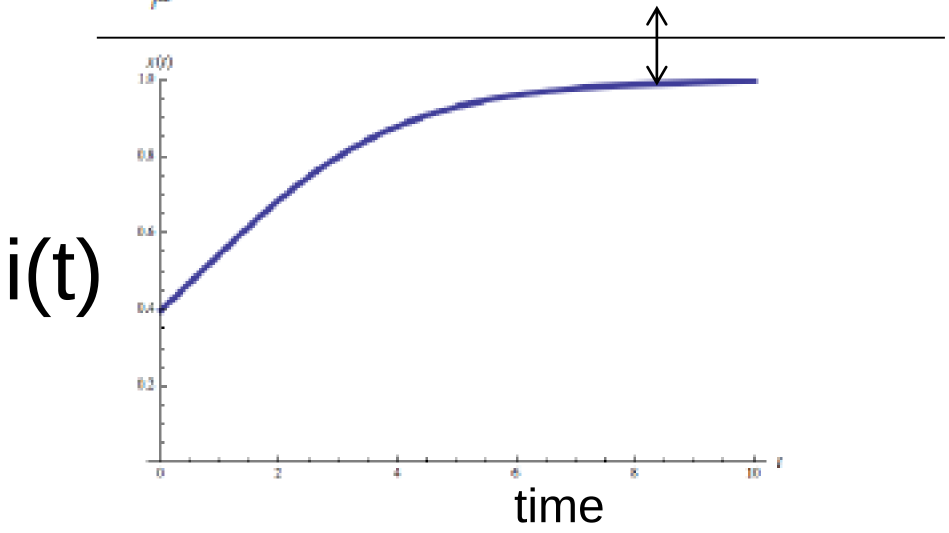
- Limit $t \to \infty$

$$\beta > \gamma \quad , \quad i(t) \to (1 - \frac{\gamma}{\beta})$$ Equilibrium condition

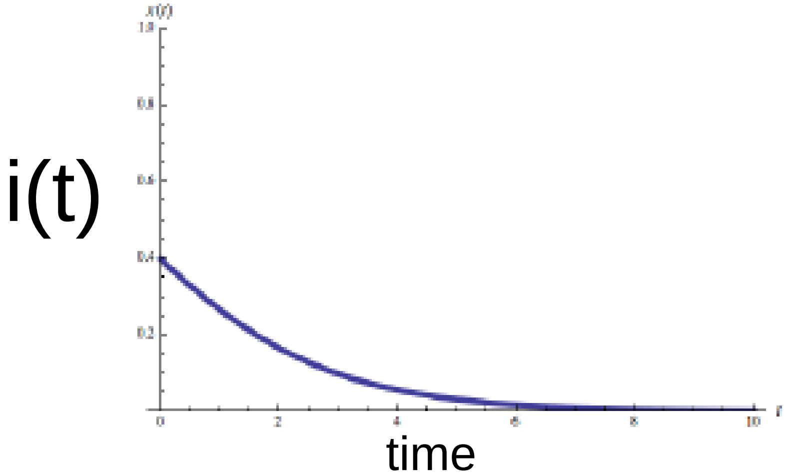
$$\beta < \gamma \quad , \quad i(t) = i_0 e^{(\beta-\gamma)t} \to 0$$

SIS model

- $\beta > \gamma, \qquad i(t) \rightarrow (1 - \frac{\gamma}{\beta})$



- $\beta < \gamma, \qquad i(t) = i_0 e^{(\beta-\gamma)t} \rightarrow 0$



SIR model

- $S(t)$ -susceptable , $I(t)$ - infected, $R(t)$ - recovered

$$S \longrightarrow I \longrightarrow R$$

$$S(t) + I(t) + R(t) = N$$

- β - infection rate, γ - recovery rate
- Infection equation:

$$\begin{aligned} \frac{ds}{dt} &= -\beta si \\ \frac{di}{dt} &= \beta si - \gamma i \\ \frac{dr}{dt} &= \gamma i \end{aligned}$$

$$s + i + r = 1$$

SIR model

- Equation

$$\frac{ds}{dt} = -\beta s \frac{dr}{dt} \frac{1}{\gamma}$$

$$s = s_0 e^{-\frac{\beta}{\gamma} r}$$

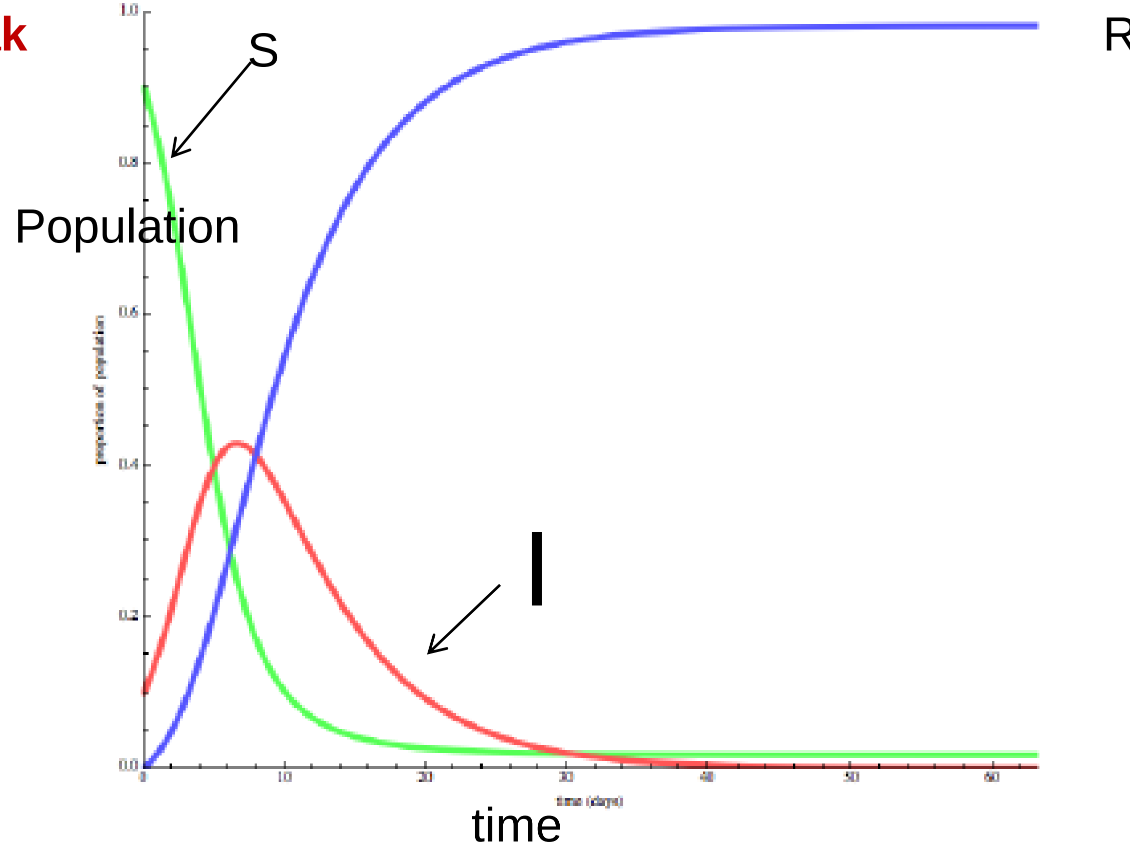
$$\frac{dr}{dt} = \gamma (1 - r - s_0 e^{-\frac{\beta}{\gamma} r})$$

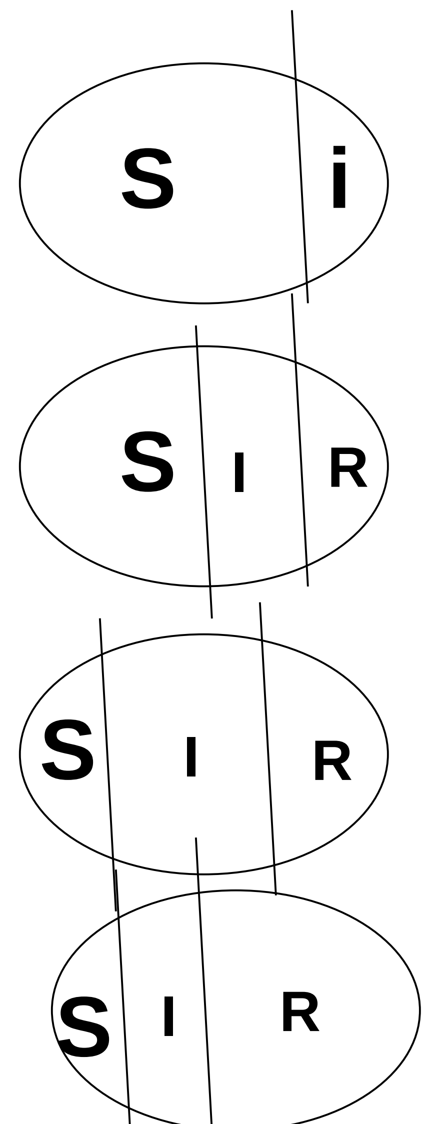
- Solution

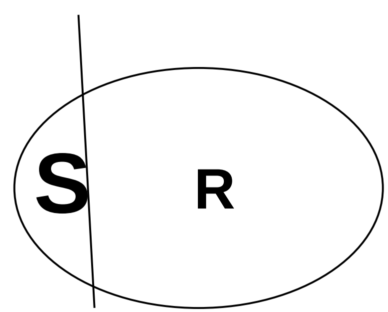
$$t = \frac{1}{\gamma} \int_0^r \frac{dr}{1 - r - s_0 e^{-\frac{\beta}{\gamma} r}}$$

SIR model

Epidemic outbreak



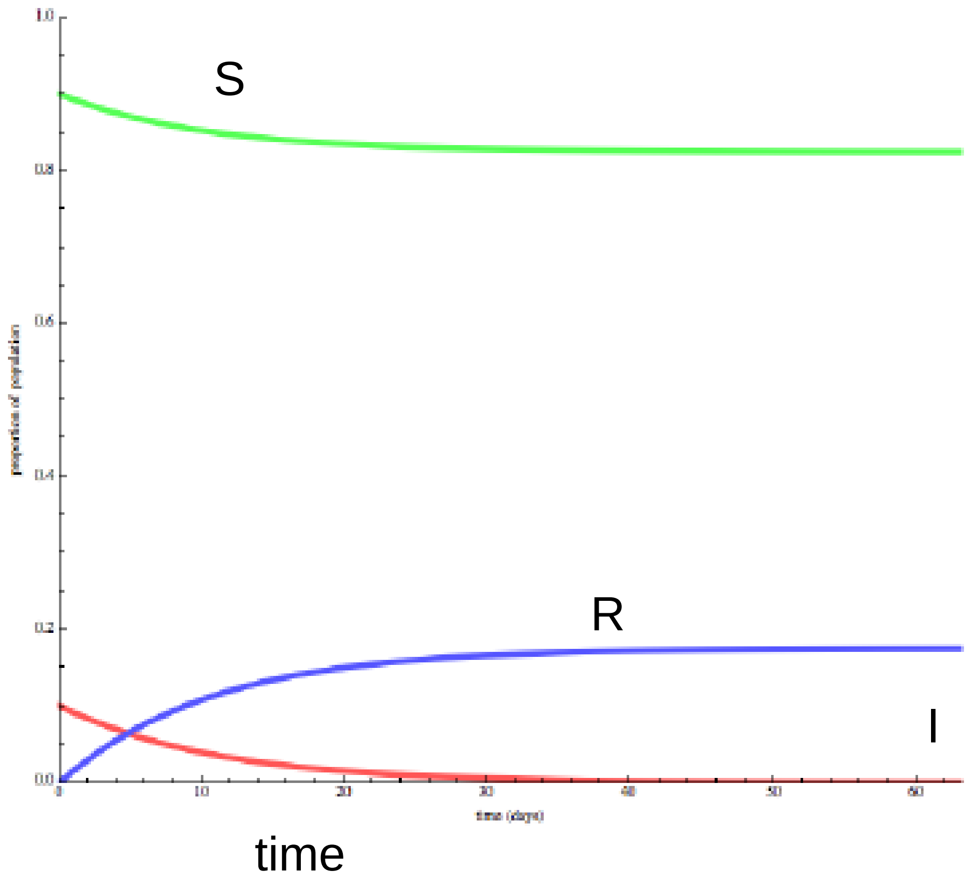




- $\frac{\beta}{\gamma} = 4$
- $i_0 = 0.1$

SIR model

No Epidemic



- $\frac{\beta}{\gamma} = 0.5$
- $i_0 = 0.1$

SIR model

- Equation

$$\frac{dr}{dt} = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma}r})$$

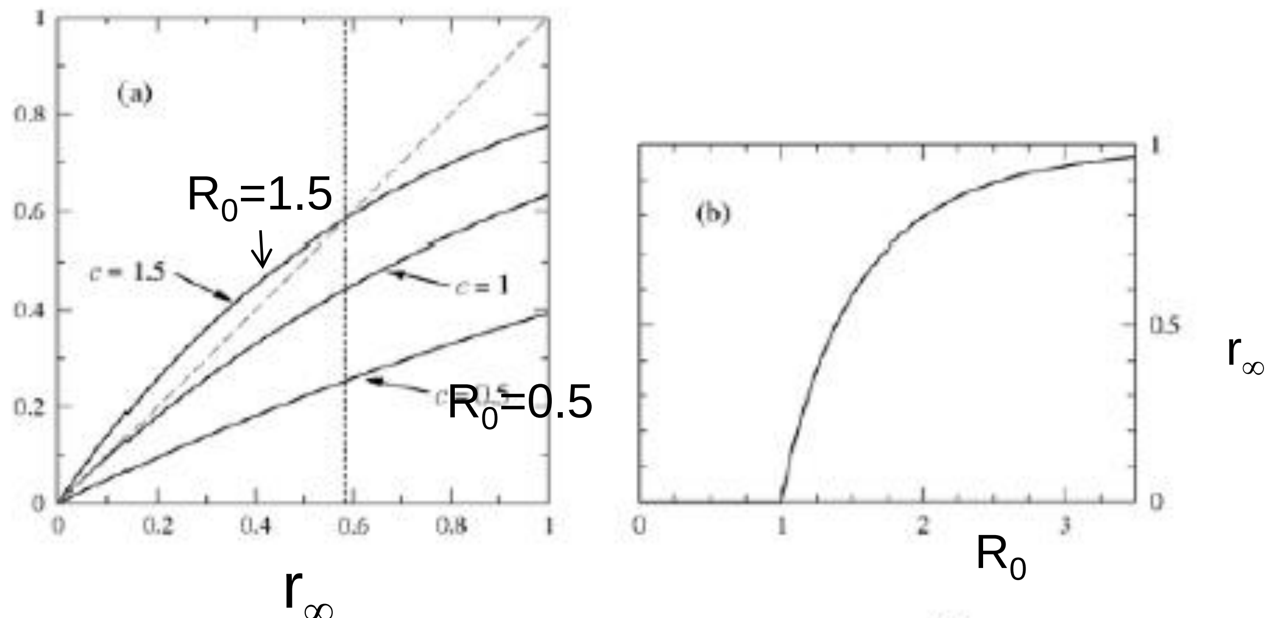
- Limits: $t \to \infty$, $\frac{dr}{dt} = 0$, $r_\infty = const$,

$$1 - r_\infty = s_0 e^{-\frac{\beta}{\gamma}r_\infty}$$

- Initial conditions: $r(0) = 0$, $i(0) = c/N$, $s(0) = 1 - c/N \approx 1$

$$1 - r_\infty = e^{-\frac{\beta}{\gamma}r_\infty}$$

SIR model



R0->∞, r_∞ ->1
R0->0, r_∞ ->0

$$r_\infty = 1 - e^{-R_0 r_\infty}, \quad R_0 = \frac{\beta}{\gamma}$$

$$(r_\infty)'|_{r_\infty=0} = (1 - e^{-R_0 r_\infty})'|_{r_\infty=0}, \quad >1$$

Non-zero solution of r_∞ exists when

critical point: $R_0 = 1$

ER graph: Phase transition

Let u – fraction of nodes that do not belong to GCC. The probability that a node does not belong to GCC

$$u = P(k=1) \cdot u + P(k=2) \cdot u^2 + P(k=3) \cdot u^3 ... =$$
$$= \sum_{k=0}^{\infty} P(k) u^k = \sum_{k=0} \frac{\lambda^k e^{-\lambda}}{k!} u^k = e^{-\lambda} e^{\lambda u} = e^{\lambda(u-1)}$$

Let s -fraction of nodes belonging to GCC (size of GCC)

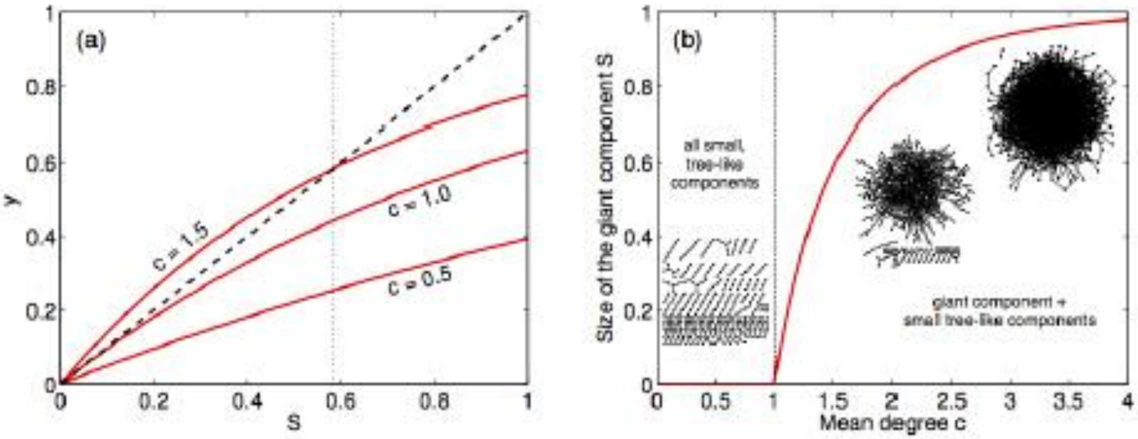
$$s = 1 - u$$

$$1 - s = e^{-\lambda s}$$

when $\lambda \to \infty, \quad s \to 1$
when $\lambda \to 0, \quad s \to 0$
$(\lambda = pn)$

ER graph: Phase transition

$$s = 1 - e^{-\lambda s}$$



non-zero solution exists when (at $s = 0$):

$$\lambda e^{-\lambda s} > 1$$

critical value:

$$\lambda_c = 1$$

$$\lambda_c = \langle k \rangle = p_c n = 1, \quad p_c = \frac{1}{n}$$

SIR model

- r_∞ - the total size of the outbreak
- Epidemic threshold

$$\begin{array}{ll} \text{Epidemics:} & R_0 > 1, \quad \beta > \gamma \quad , \quad r_\infty = const > 0 \\ \text{No epidemics:} & R_0 < 1, \quad \beta < \gamma \quad , \quad r_\infty \to 0 \end{array}$$

- Basic reproduction number

$$R_0 = \frac{\beta}{\gamma}$$