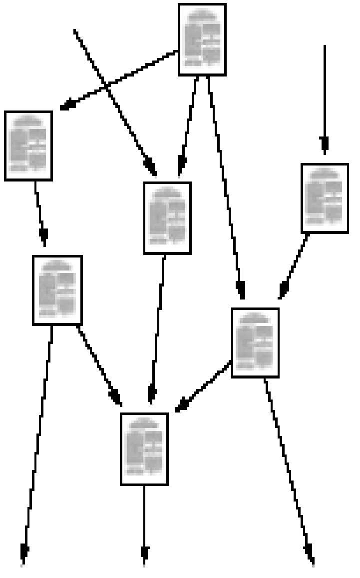
Network Analysis

Degree Distribution: The case of Citation Networks

- **Papers (in almost all fields) refer to works done earlier on same/related topics – *Citations***
- **A network can be defined as**
  - **Each node is a paper**
  - **A directed edge from paper A to paper B indicates A cites B**
- **These networks are acyclic**
- **Edges point backward in time!**



citation network

Consider nodes as researchers and links as citations

Law of Scientific Productivity

- **Alfred Lotka (1926) did some analysis of such a citation network and made a statement**
  - ***the number of scientists who have k citations falls off as $k^{-\alpha}$ for some constant α.***
- **Considering each node in the citation network to be representative of scientists can you say what exactly did Lotka study???**

The distribution of the degree of the nodes !!!

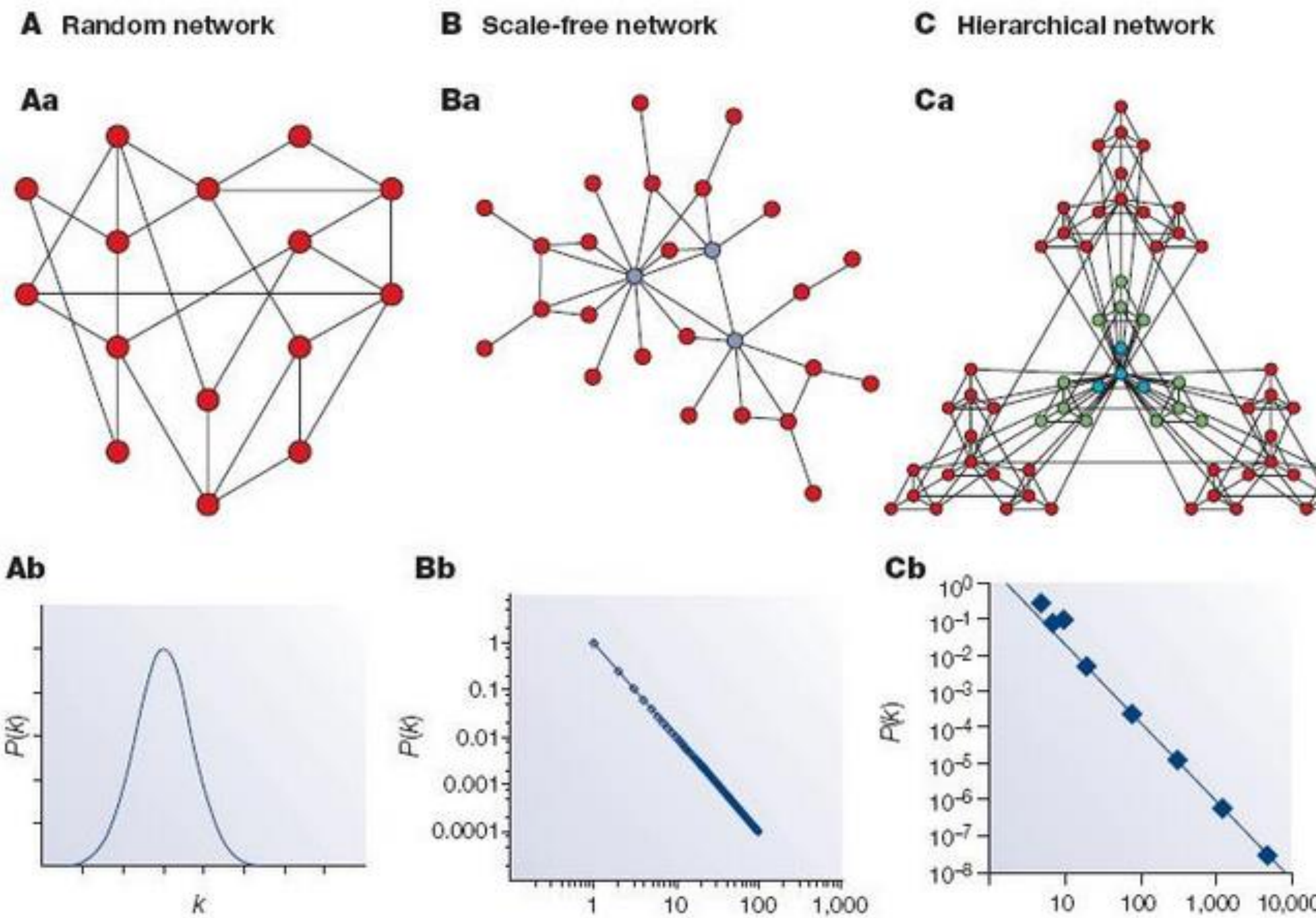
Degree Distribution: Formal Definition

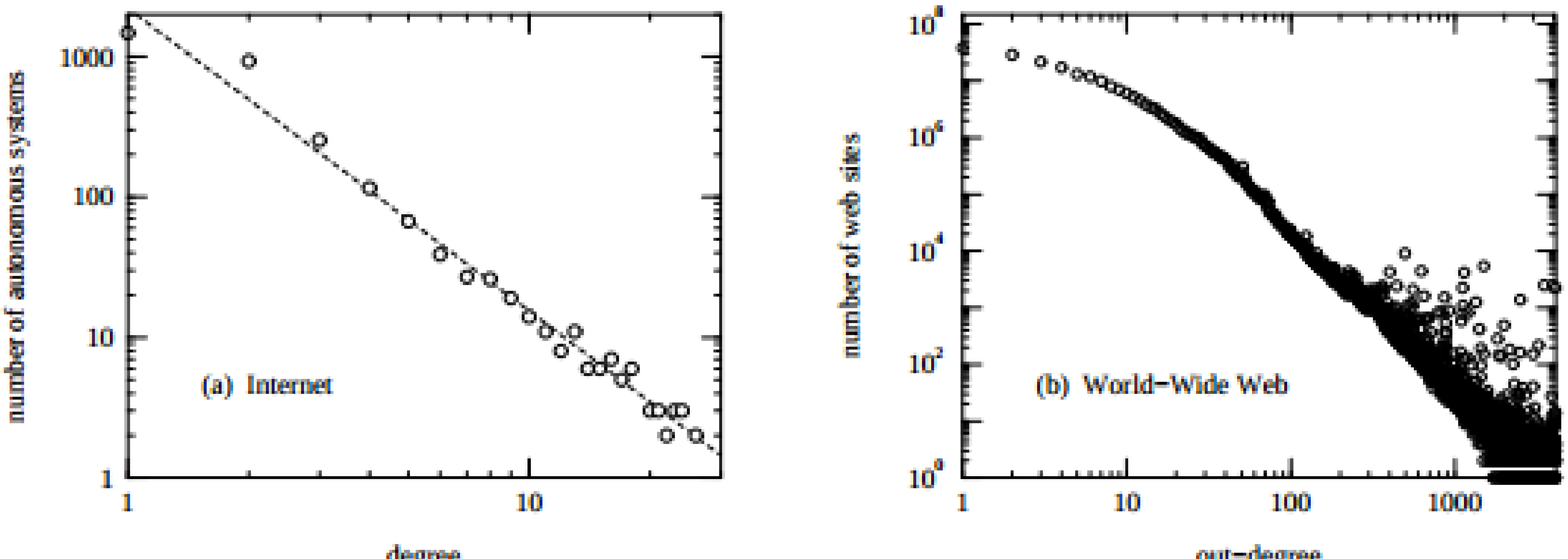
- **Let p_k be the fraction of vertices in the network that has a degree k**

- **Hence p_k is the probability that a vertex chosen uniformly at random has a degree k**

- **The k versus p_k plot is defined as the degree distribution of a network**

- **For most of the real world networks these distributions are right skewed with a long right tail showing up values far above the mean – p_k varies as $k^{-\alpha}$**


A Random network
B Scale-free network
C Hierarchical network
Aa
Ba
Ca
Ab
Bb
Cb
P(k)
k
1
0.1
0.01
0.001
0.0001
1
10
100
1,000
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10
100
1,000
10,000


number of autonomous systems
1000
100
10
1
1
10
degree
(a) Internet
number of web sites
10^8
10^6
10^4
10^2
10^0
1
10
100
1000
out−degree
(b) World−Wide Web

The Definition Slightly Modified

- **Due to noisy and insufficient data sometimes the definition is slightly modified**
  - **Cumulative degree distribution is plotted**

$$P_k = \sum_{k'=k}^{\infty} p_{k'},$$

- **Probability that the degree of a node is greater than or equal to k**

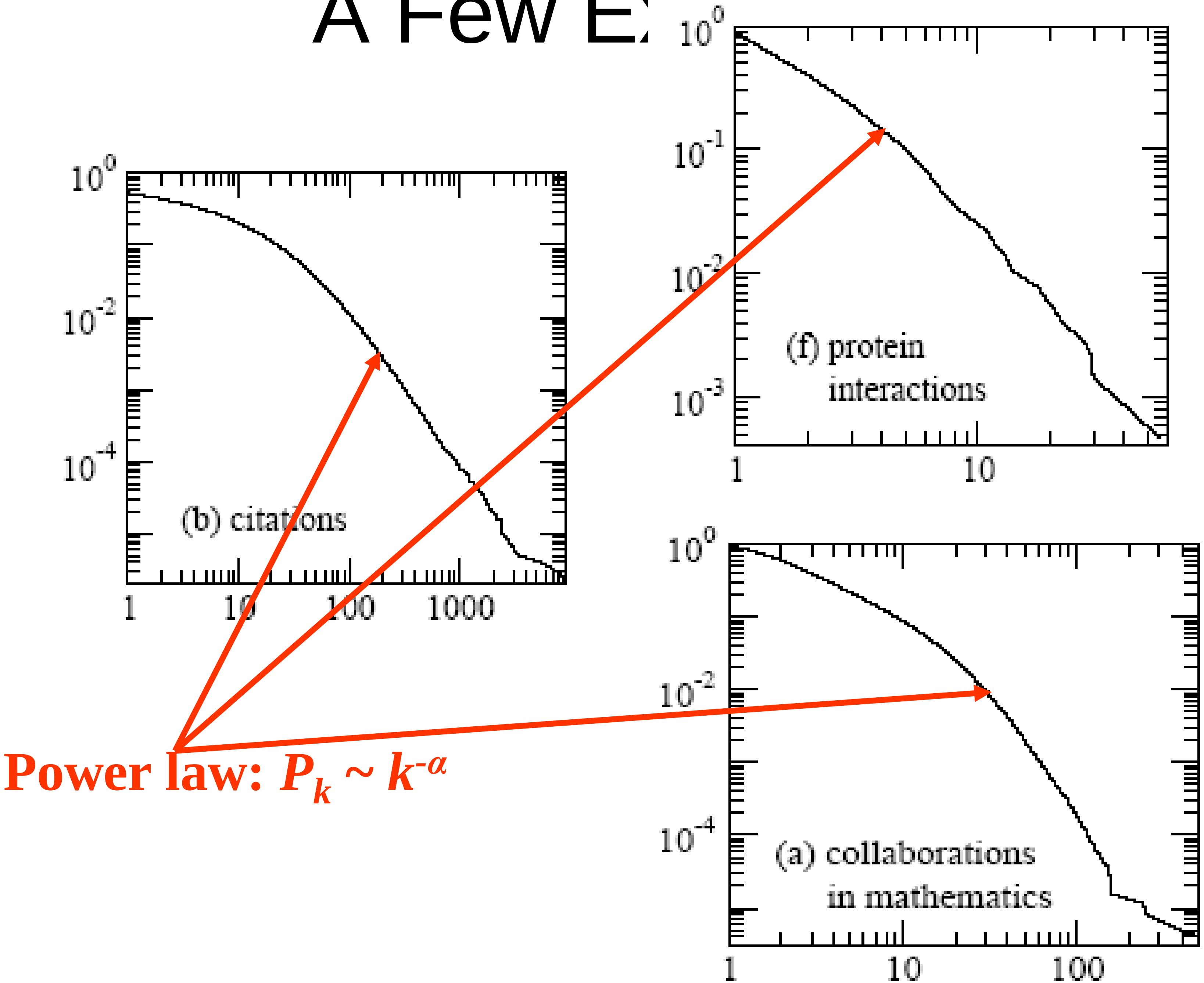
$$p_k \sim k^{-\alpha}$$

$$P_k = \int_{k'=k}^{\infty} p_{k'} dk'$$

Continuous scale

$$p_k \sim \mathrm{e}^{-k/\kappa}.$$

A Few E



Emerging property

Scale-free

For any function $f(x)$

the independent variable when rescaled $f(ax)$

does not affect the functional form $bf(x)$

$$f(ax) = bf(x)$$

Power-laws – are they scale-free???

Geodesic distance

Mean distance between the pair of nodes

$$\ell = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}$$

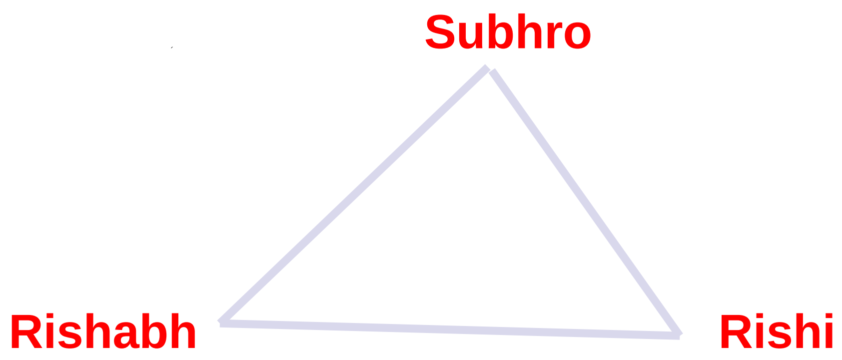
Alternate measure (harmonic mean)

$$\ell^{-1} = \frac{1}{\frac{1}{2}n(n+1)} \sum_{i \geq j} d_{ij}^{-1}.$$

network	type	n	m	z	ℓ
film actors	undirected	449 913	25 516 482	113.43	3.48
company directors	undirected	7 673	55 392	14.44	4.60
math coauthorship	undirected	253 339	496 489	3.92	7.57
physics coauthorship	undirected	52 909	245 300	9.27	6.19
biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92
telephone call graph	undirected	47 000 000	80 000 000	3.16	
email messages	directed	59 912	86 300	1.44	4.95
email address books	directed	16 881	57 029	3.38	5.22
student relationships	undirected	573	477	1.66	16.01
sexual contacts	undirected	2 810			
WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27
WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18
citation network	directed	783 339	6 716 198	8.57	
Roget's Thesaurus	directed	1 022	5 103	4.99	4.87
word co-occurrence	undirected	460 902	17 000 000	70.13	
Internet	undirected	10 697	31 992	5.98	3.31
power grid	undirected	4 941	6 594	2.67	18.99
train routes	undirected	587	19 603	66.79	2.16

Friend of Friends are Friends

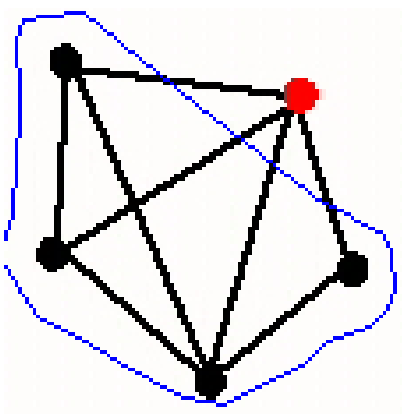
- **Consider the following scenario**
  - **Subhro and Rishabh are friends**
  - **Rishabh and Rishi are friends**
  - **Are Subhro and Rishi friends?**
  - **If so then …**

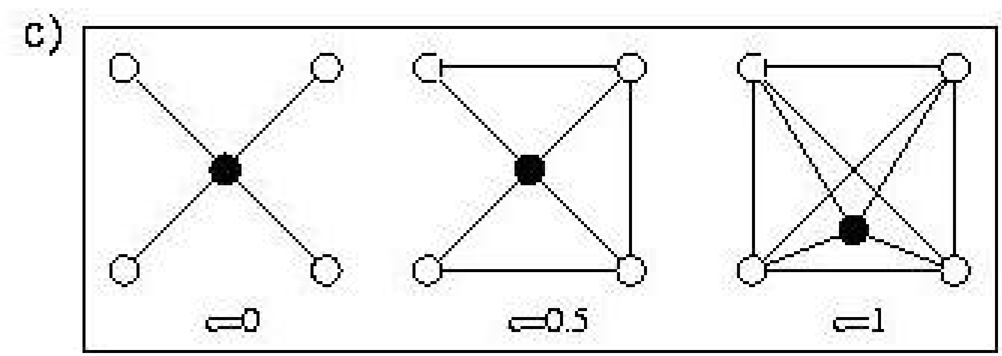


- **This property is known as transitivity**

Measuring Transitivity: Clustering Coefficient

- **The clustering coefficient for a vertex '*v*' in a network is defined as the ratio between the total number of connections among the neighbors of '*v*' to the total number of possible connections between the neighbors**





- **The philosophy – High clustering coefficient means my friends know each other with high probability – a typical property of social networks**

Mathematically…

- The clustering index of a vertex *i* is

$$C_i = \frac{\text{\# of links between neighbors}}{n(n-1)/2}$$

- The clustering index of the whole network is the average

$$C = \frac{1}{N} \sum C_i$$

Network	C	C_{rand}	L	N
WWW	0.1078	0.00023	3.1	153127
Internet	0.18-0.3	0.001	3.7-3.76	3015-6209
Actor	0.79	0.00027	3.65	225226
Coauthorship	0.43	0.00018	5.9	52909
Metabolic	0.32	0.026	2.9	282
Foodweb	0.22	0.06	2.43	134
C. elegance	0.28	0.05	2.65	282

Local definition

Global definition

$$C = \frac{3\times \text{ number of triangles in the network}}{\text{number of connected triples of vertices}},$$

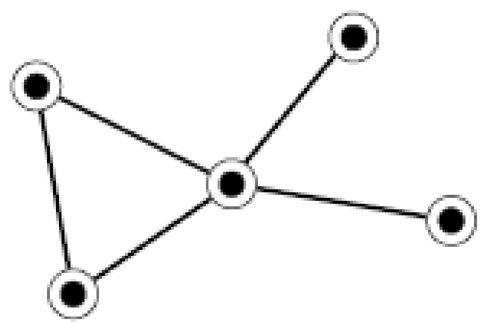


FIG. 5 Illustration of the definition of the clustering coefficient C, Eq. (3). This network has one triangle and eight connected triples, and therefore has a clustering coefficient of $3 \times 1/8 = \frac{3}{8}$. The individual vertices have local clustering coefficients, Eq. (5), of 1, 1, $\frac{1}{6}$, 0 and 0, for a mean value, Eq. (6), of $C = \frac{13}{30}$.

Connected triplet is defined to be a connected subgraph consisting of three vertices and two edges.

"connected triple" means a single vertex with edges running to an unordered pair of others

The World is Small!

- All late registrants in the Complex Networks course shall get 10 marks bonus!!!!!
- How long do you think the above information will take to spread among yourselves
- Experiments say it will spread very fast – within 6 hops from the initiator it would reach all
- This is the famous Milgram’s six degrees of separation

Milgram's Experiment

- Travers & Milgram 1969: classic study in early social science
  - **Source stockbrokers**
  - **Destination stockbroker**

  - **Job: Forward a letter to a friend "closer" to the target**
  - **Target information provided:**
    - **name, address, occupation, firm, college, wife's name and hometown**

Milgram typically chose individuals in the U.S. cities of Omaha, Nebraska, and Wichita, Kansas, to be the starting points and Boston, Massachusetts, to be the end point

Findings

- Most of the letters in this experiment were lost…
- Nevertheless a quarter reached the target
- Strikingly those that reached the target passed through the hands of

six people on an average

- In fact

- 64 of 296 chains reached the target

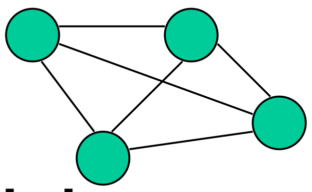
- average length of *completed* chains: 5.2

Structural Holes and Redundancy

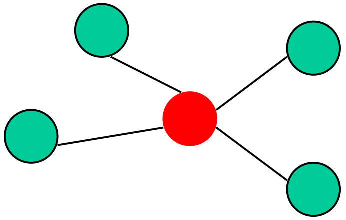
Structural holes are nodes (mainly in a social network) that separate non-redundant sources of information------------ sources that are additive than overlapping

Redundancy

- **Cohesion – contacts strongly connected to each other are likely to have similar information and therefore provide redundant information benefits**

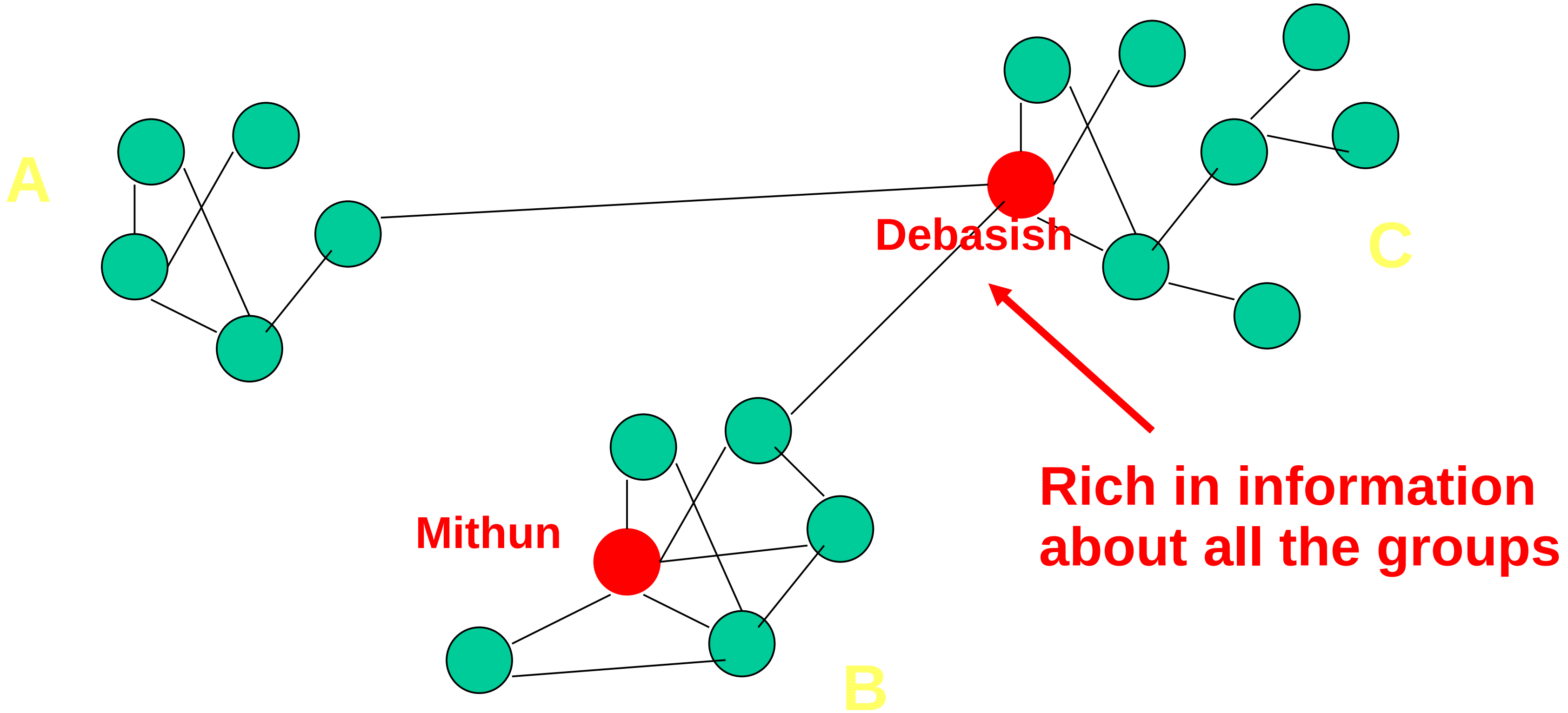


- **Equivalence – contacts that link a manager to the third parties……. have same sources of information (manager) and therefore provide redundant information benefits**



Structural Holes broker Information

Consider the following network



Debasish has the opportunity to play a information broker – but Mithun doesn't

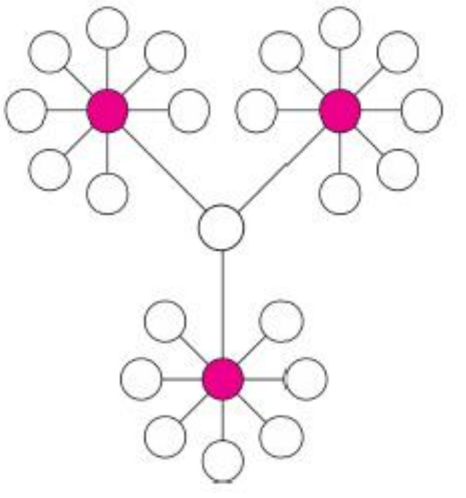
Centrality

Which vertices are important?

Centrality

- Centrality measures are commonly described as indices of

  - prestige,
  - prominence,
  - importance,
  - and power -- the four I



- A measure indicating the importance of a vertex

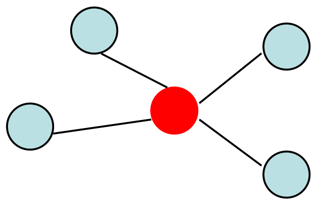
Degree Centrality

Degree Centrality – Immediate neighbors of a vertex (k) expressed as a fraction of the total number of neighbors possible

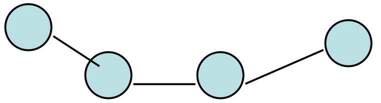
$$\text{Degree Centrality} = \frac{k}{N-1}$$

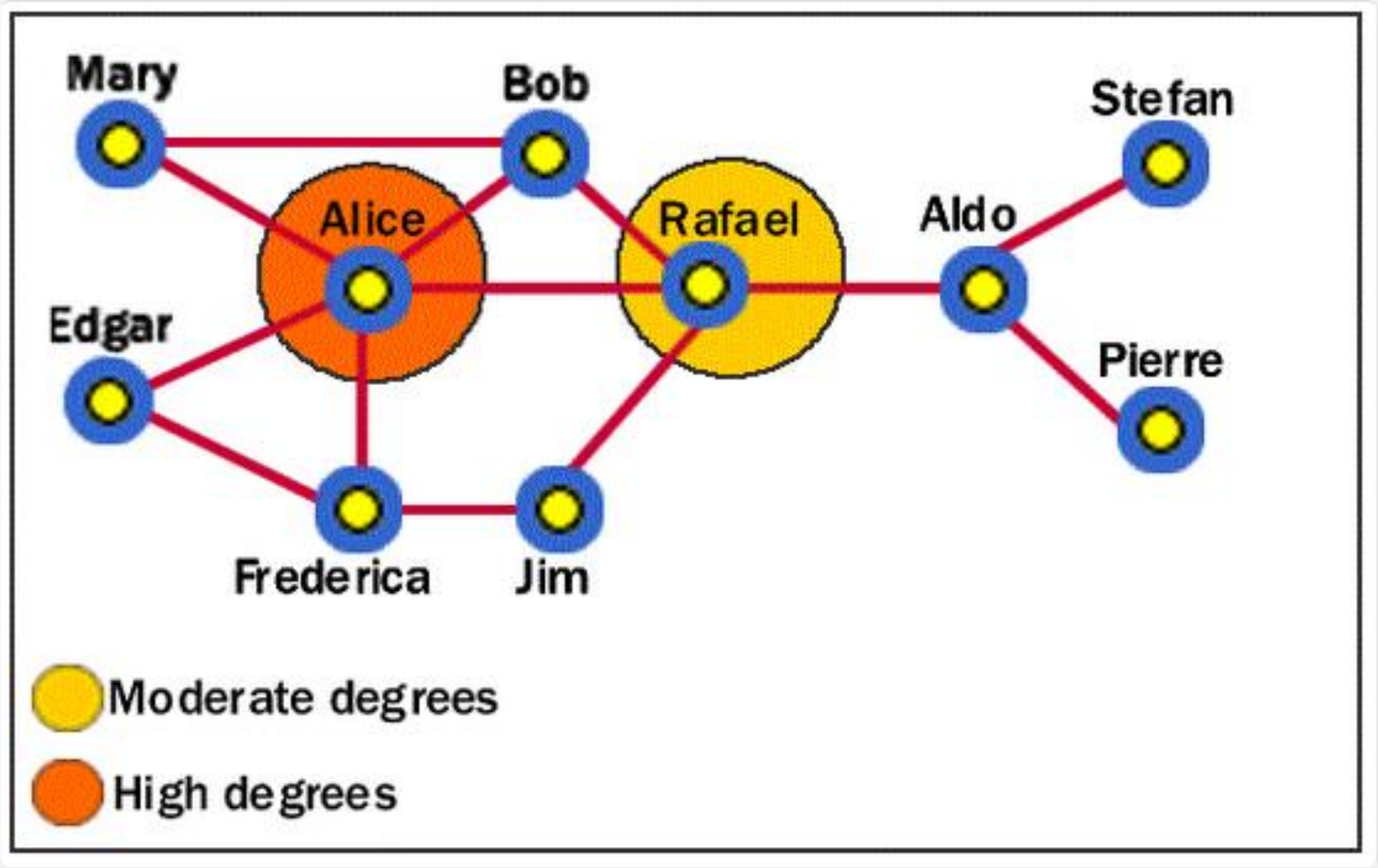
Variance of degree centrality – Centralization

Star network – an ideal centralized one



Line network – less centralized




Mary
Bob
Stefan
Alice
Rafael
Aldo
Edgar
Pierre
Frederica
Jim
Moderate degrees
High degrees

Closeness centrality

Closeness centrality (or closeness) of a node v

Average length of the shortest path between the node v and all other nodes in the graph.

$$C_C(i) = \frac{1}{\sum_j d(i,j)}$$

Thus the more central a node is, the closer it is to all other nodes.

Harmonic centrality $= \sum_j \frac{1}{d(i,j)}$

Normalized closeness centrality

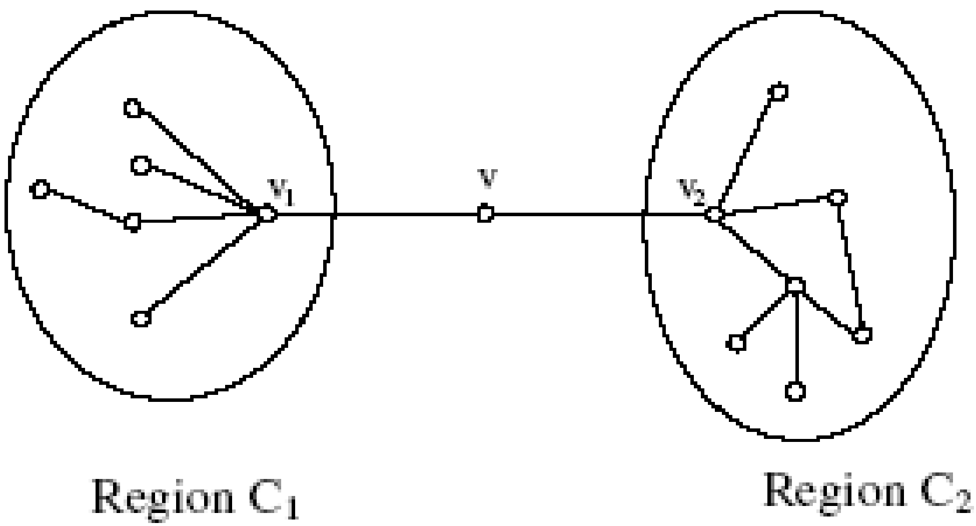
$$C_C^*(i) = (n-1)C_C(i)$$

Betweenness Centrality

- Tries to determine how important is a node in a network
- Degree of a node doesn't only determine its importance in the network – do you agree???

Betweenness Centrality

- Tries to determine how important is a node in a network
- Degree of a node doesn't only determine its importance in the network – do you agree???
- The node can be on a *bridge* centrally between two regions of the network!!



Betweenness Centrality'

- Centrality of *v*: Ratio: the number of shortest paths that pass through v Vs total number of shortest paths from node s to node t.

$$g(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

Normalized betweenness centrality

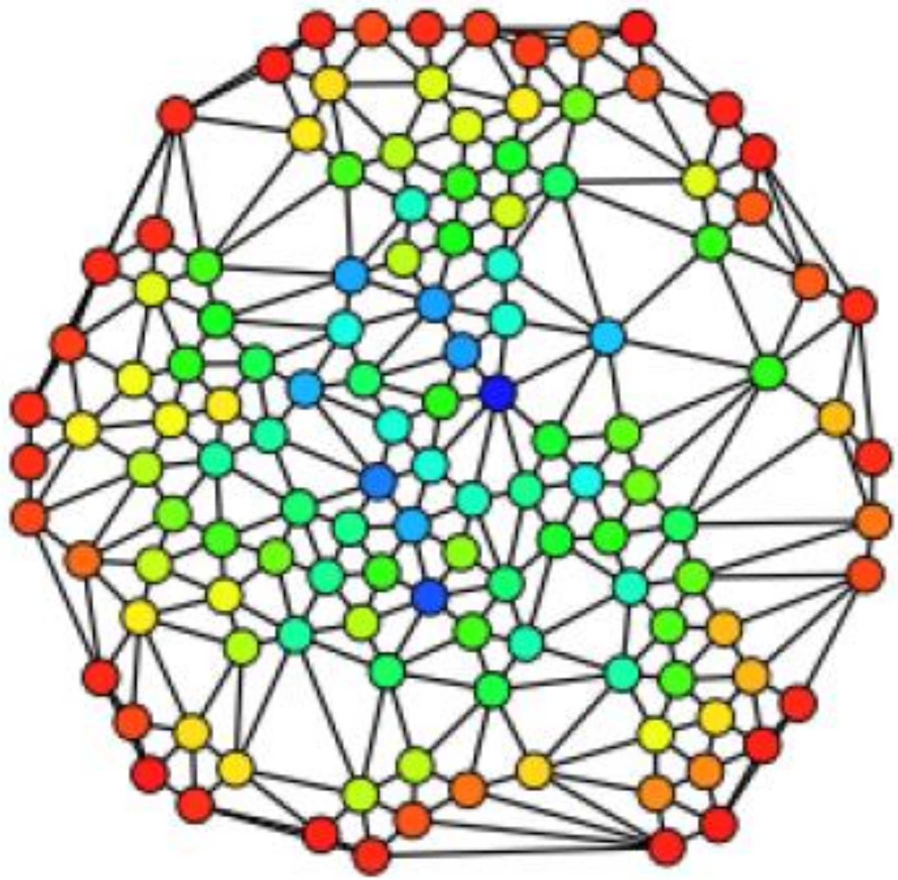
$$C_B^*(i) = \frac{2}{(n-1)(n-2)} C_B(i)$$

Betweenness Centrality

- Removal – what can this lead to??

Betweenness Centrality

- Removal – what can this lead to??
- Increase in the geodesic path – extreme case is infinity (network gets disconnected)
- Can you visualize the impact of removal of the nodes with high betweenness in the following networks??
  - Epidemic network
  - Information network
  - Traffic network



Hue (from red=0 to blue=max) shows the node betweenness.

Flow Betweenness

- What if the nodes with high betweenness behave as reluctant brokers and do not allow two other nodes (of different regions) to establish a relationship.
- They must find other ways to establish relationship (may not be cost effective)
  - Something like "wanting to propose someone via a third party (say his/her friends) who is also (kind of) your friend – but this common friend is reluctant to pursue the proposal!"
- This is the main idea of flow betweenness

Flow Betweenness

- Let m_{jk} be the amount of flow between vertex j and vertex k which must pass through i for any maximum flow.

- The flow betweenness of vertex i is the sum of all m_{jk} where i, j and k are distinct and $j < k$.

    - The flow betweenness is therefore a measure of the contribution of a vertex i to all possible maximum flows.

- The normalized flow betweenness centrality of a vertex i is the flow betweenness of i divided by the total flow through all pairs of points where i is not a source or sink.

- Takes into account all paths (not only the shortest ones) from j to *k* via *i* – computationally quite intractable for large networks.

Eigenvector Centrality (Bonacich 1972)

- In context of HIV transmission – A person x with one sex partner is less prone to the disease than a person y with multiple partners

Eigenvector Centrality (Bonacich 1972)

- In context of HIV transmission – A person x with one sex partner is less prone to the disease than a person y with multiple partners
- But imagine what happens if the partner of x has multiple partners
- It is not just how many people know me counts to my popularity (or power) but how many popular people know me – this is recursive!
- The basic idea of eigenvector centrality

Eigenvector Centrality

- Idea is to define centrality of vertex as sum of centralities of neighbors.
- Suppose we guess initially vertex i has centrality $x_{\boldsymbol{i}}(0)$
- Improvement is $x_{\boldsymbol{i}}(1) \leftarrow \sum_j A_{\boldsymbol{i}j}\, x_{\boldsymbol{j}}(0)$
- Continue until there is no more improvement observed
- So, $x(t) \leftarrow \mathbf{A}x(t\text{-}1)$ => $x(t) \longleftarrow \mathbf{A}^t x(0)$ [Power iteration method proposed by Hotelling]

$$\mathbf{A}^t x = \lambda^t x$$

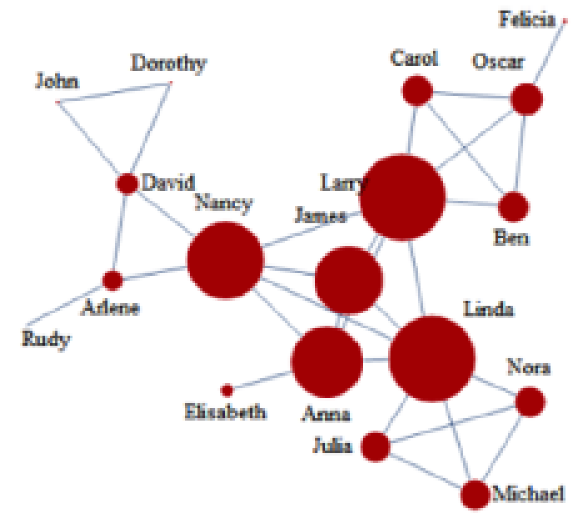
Eigenvector Centrality (Bonacich 1972)

Importance of a node depends on the importance of its neighbors (recursive definition)

$$v_i \leftarrow \sum_j A_{ij} v_j$$

$$v_i = \frac{1}{\lambda} \sum_j A_{ij} v_j$$

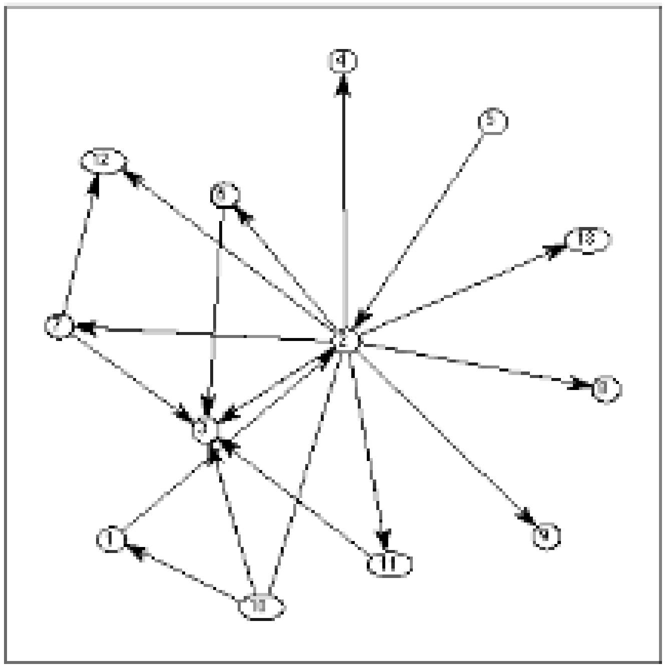
$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

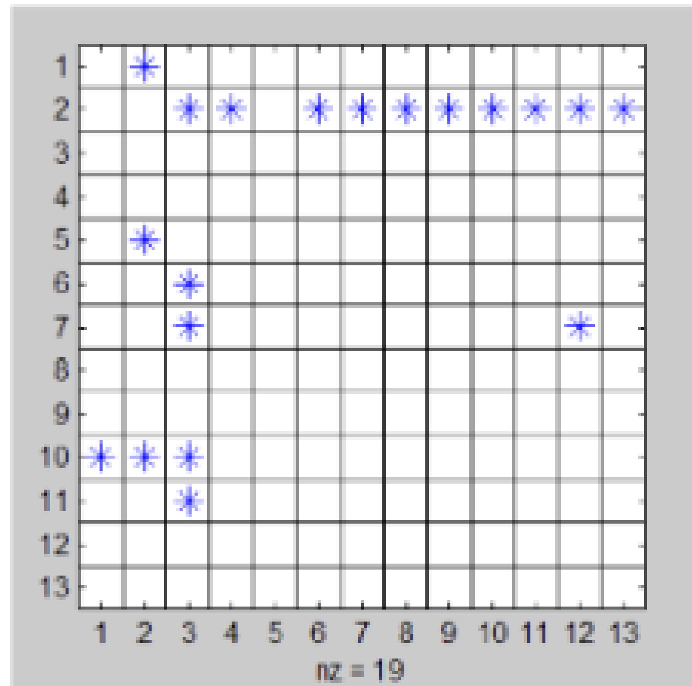


Select an eigenvector associated with largest eigenvalue $\lambda = \lambda_1$, $\mathbf{v} = \mathbf{v}_1$

Adjacency matrix and eigenvector

Graph $G(n, m)$, adjacency matrix A_{ij}, edge $i \to j$





Eigenvalue

$$A\mathbf{v} = \lambda\mathbf{v}.$$

2D matrix Eigenvector

But how to find it –
Power Accelerated Method of Hotelling

Step 0. Set $e_i = 1$ for all i.

Step 1. Compute $e_i^* = \sum_j a_{ij} e_j$.

Step 2. Set λ equal to the square root of the sum of squares of e^*.

Step 3. Set $e_i = e_i^*/\lambda$ for all i.

Step 4. Repeat steps 1 to 3 until λ stops changing.

- **Note that after executing step 1 the first time, e* is equal to simple degree.**

Example 3.3. *For the graph shown in Figure 3.2(b), the adjacency matrix is as follows:*

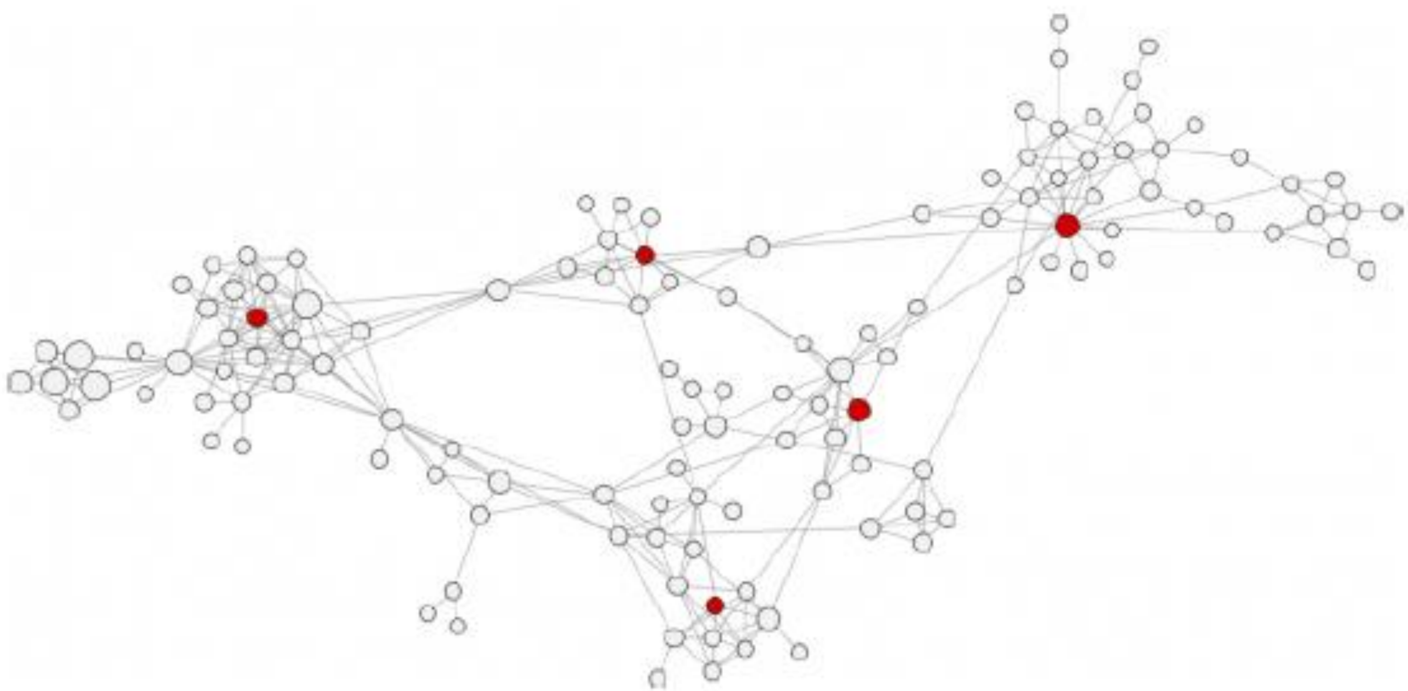
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.17}$$

The eigenvalues of A are $(-1.74, -1.27, 0.00, +0.33, +2.68)$. *For eigenvector centrality, the largest eigenvalue is selected:* 2.68. *The corresponding eigenvector is the eigenvector centrality vector and is*

$$\mathbf{C}_e = \begin{bmatrix} 0.4119 \\ 0.5825 \\ 0.4119 \\ 0.5237 \\ 0.2169 \end{bmatrix}. \tag{3.18}$$

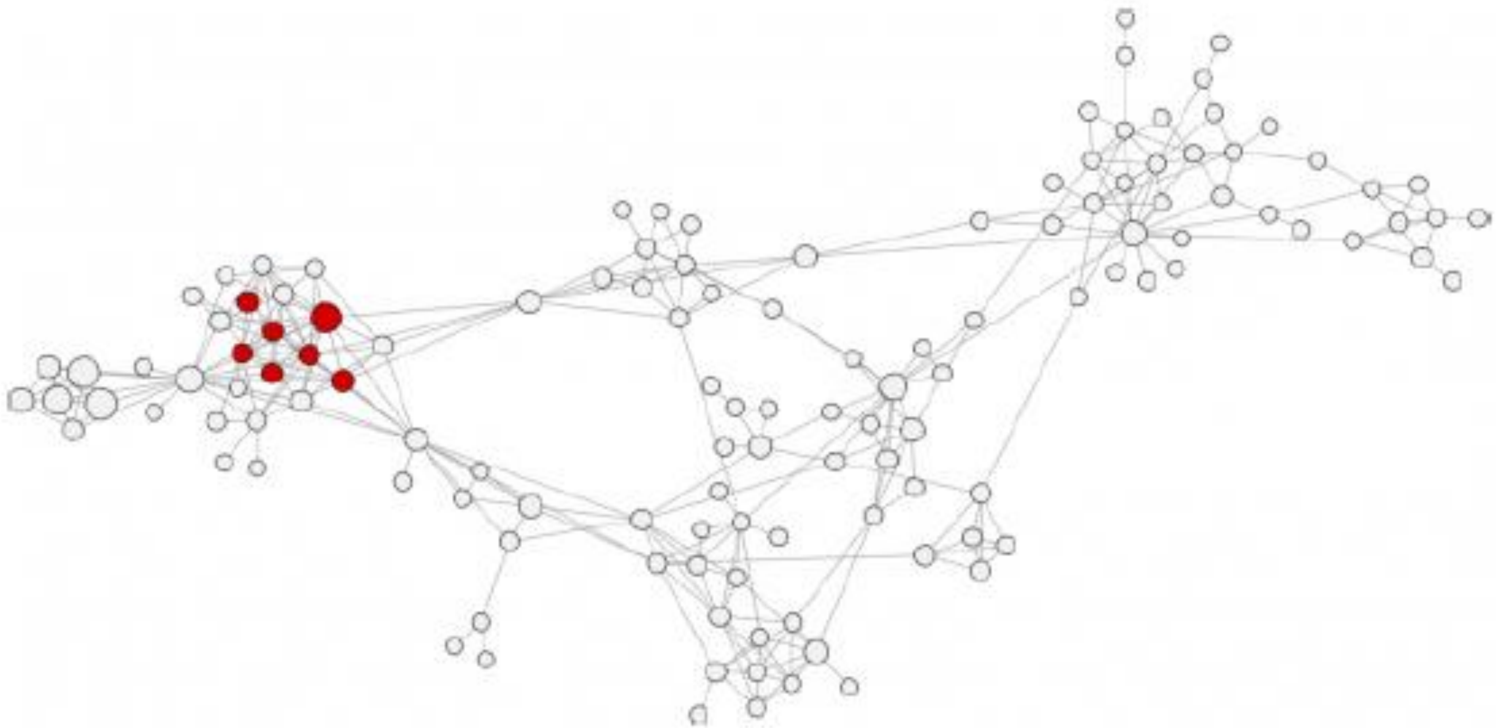
Based on eigenvector centrality, node v_2 is the most central node.

Closeness centrality



Betweenness centrality

Eigenvector centrality



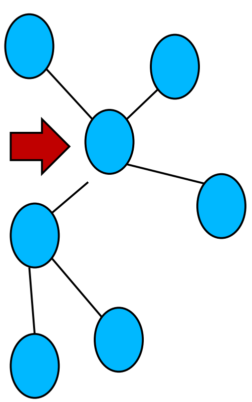
Katz Centrality

Weighted count of all paths coming to the node: the weight of path of length n is counted with attenuation factor β^n, $\beta < \frac{1}{\lambda_1}$

$$k_i = \beta \sum_j A_{ij} + \beta^2 \sum_j A_{ij}^2 + \beta^3 \sum_j A_{ij}^3 + \ldots$$

$$\mathbf{k} = (\beta \mathbf{A} + \beta^2 \mathbf{A}^2 + \beta^3 \mathbf{A}^3 + \ldots)\mathbf{e} = \sum_{n=1}^{\infty} (\beta^n \mathbf{A}^n)\mathbf{e} = (\sum_{n=0}^{\infty} (\beta \mathbf{A})^n - \mathbf{I})\mathbf{e}$$

$$\sum_{n=0}^{\infty} (\beta \mathbf{A})^n = (\mathbf{I} - \beta \mathbf{A})^{-1}$$



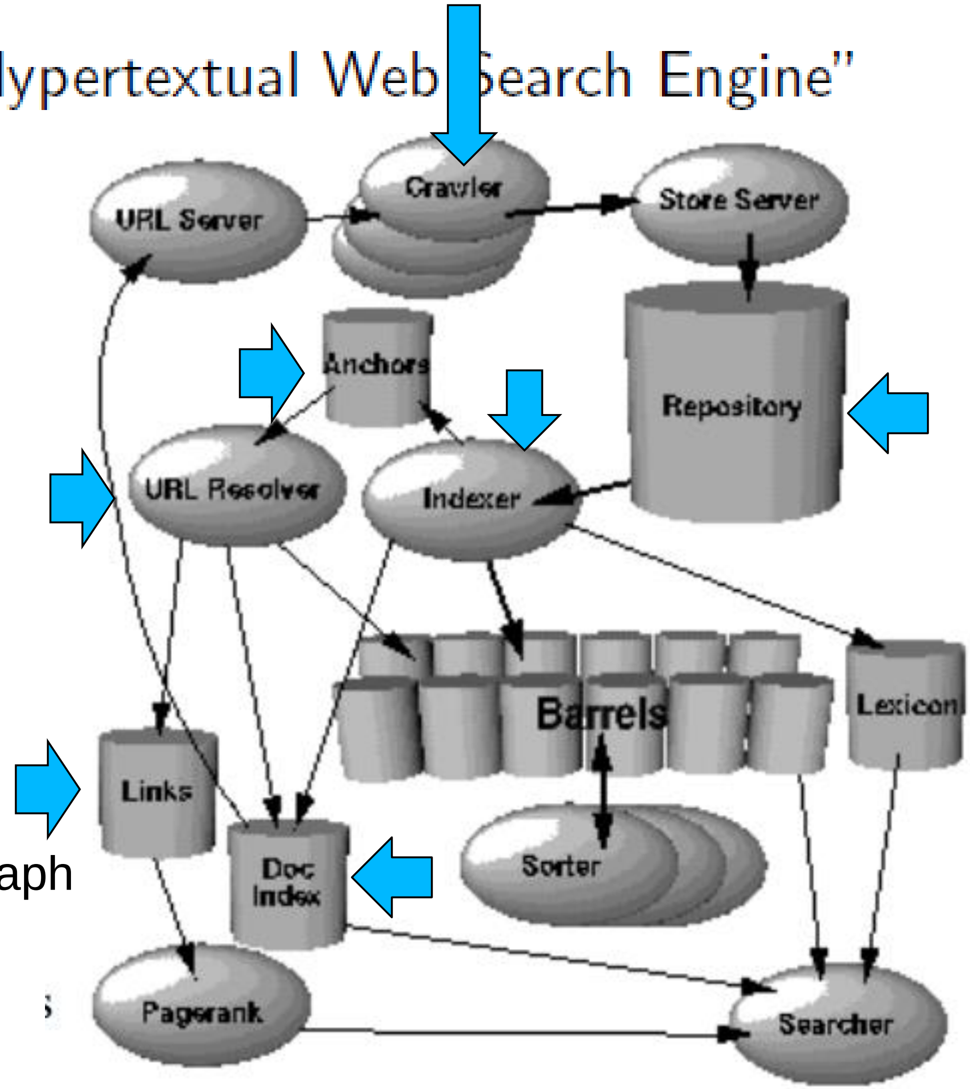
$$\mathbf{k} = ((\mathbf{I} - \beta \mathbf{A})^{-1} - \mathbf{I})\mathbf{e}$$

$$(\mathbf{I} - \beta \mathbf{A})\mathbf{k} = \beta \mathbf{A}\mathbf{e}$$

$$\mathbf{k} = \beta (\mathbf{I} - \beta \mathbf{A})^{-1} \mathbf{A}\mathbf{e}$$

Pagerank

"The Anatomy of a Large-Scale Hypertextual Web Search Engine"



Web graph

Sergey Brin and Lawrence Page, 1998

Pagerank

The Anatomy of a Large-Scale Hypertextual Web Search Engine

Sergey Brin and Lawrence Page

Computer Science Department,
Stanford University, Stanford, CA 94305, USA
sergey@cs.stanford.edu and page@cs.stanford.edu

Abstract

In this paper, we present Google, a prototype of a large-scale search engine which makes heavy use of the structure present in hypertext. Google is designed to crawl and index the Web efficiently and produce much more satisfying search results than existing systems. The prototype with a full text and hyperlink database of at least 24 million pages is available at http://google.stanford.edu/
To engineer a search engine is a challenging task. Search engines index tens to hundreds of millions of web pages involving a comparable number of distinct terms. They answer tens of

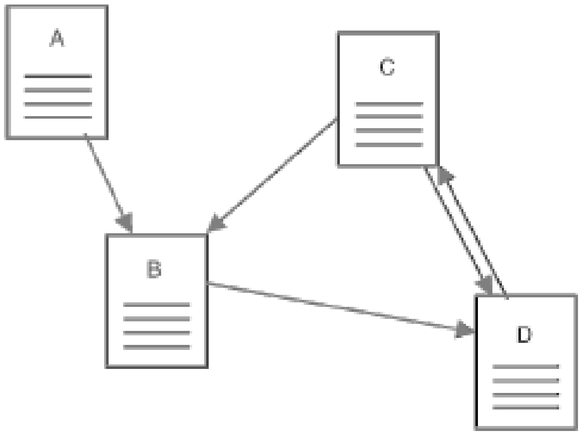
Pagerank

- Hyperlinks - implicit endorsements

citations

- Web graph - graph of endorsements (sometimes reciprocal)

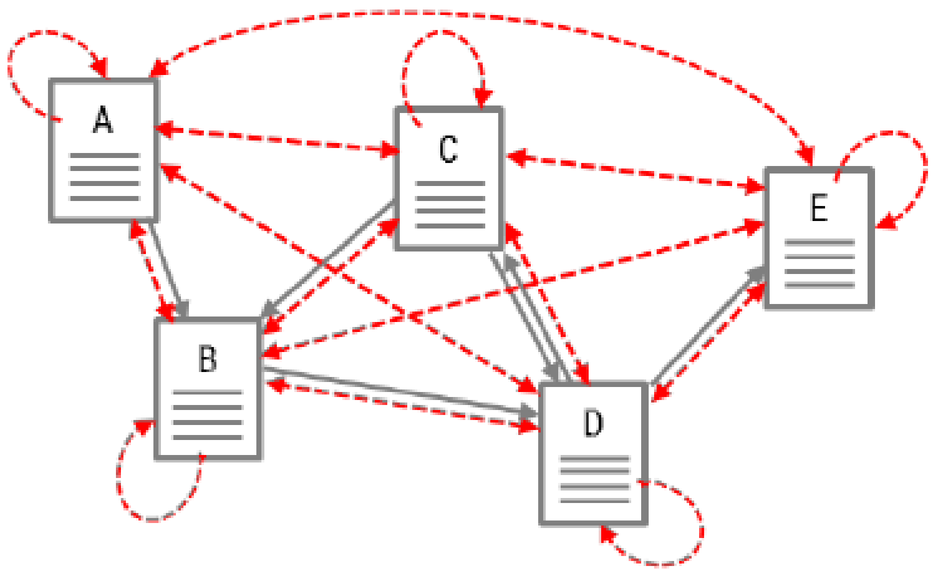
Spam the engine---link farm!!



Pagerank

"PageRank can be thought of as a model of user behavior. We assume there is a "random surfer" who is given a web page at random and keeps clicking on links, never hitting "back" but eventually gets bored and starts on another random page. The **probability** that the random surfer visits a page is its **PageRank**."

$$PR(A) = (1 - d) + d(PR(T1)/C(T1) + ... + PR(Tn)/C(Tn))$$



Random walk on graph

Transition matrix

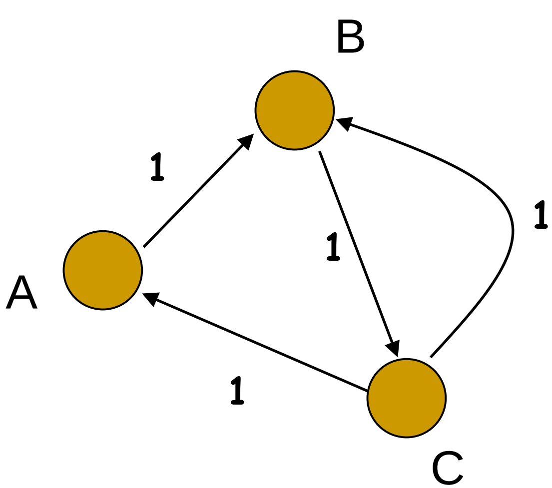
$$
\begin{array}{c|ccc}
 & A & B & C \\
\hline
A & 0 & 1 & 0 \\
B & 0 & 0 & 1 \\
C & 1 & 1 & 0
\end{array}
\qquad\qquad
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1/2 & 1/2 & 0
\end{bmatrix}
$$

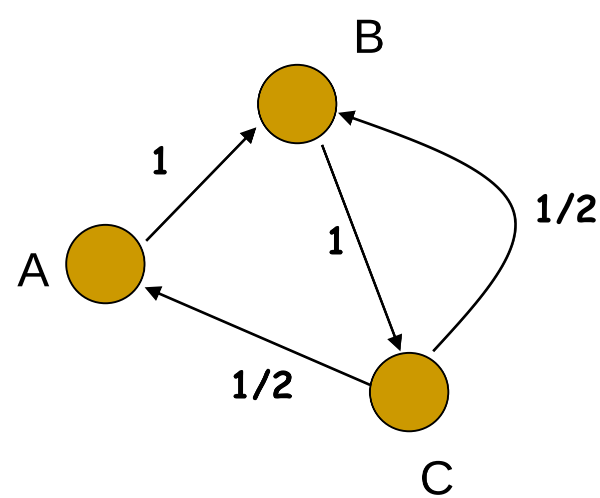
Only probabilities
--> stochastic
matrix

Adjacency matrix A

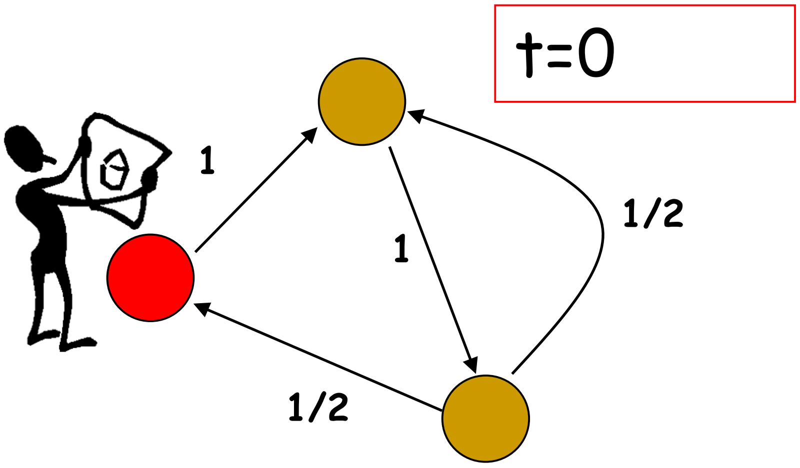
Transition matrix P

$A_{ij}/\sum_{\mathbf{i}} A_{ij}$

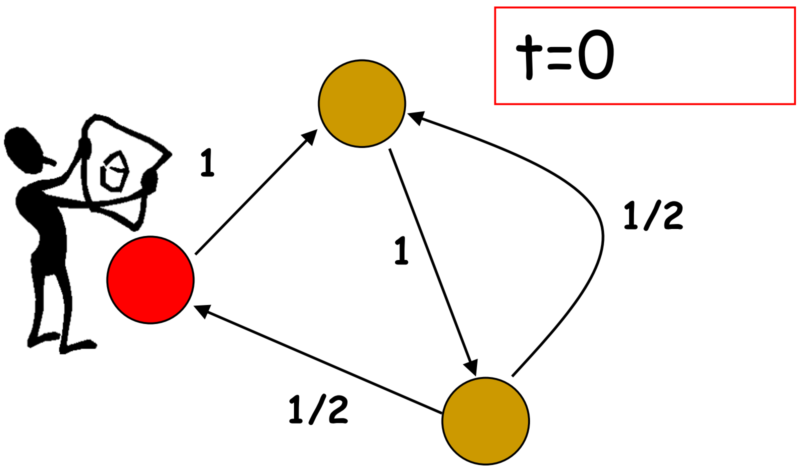


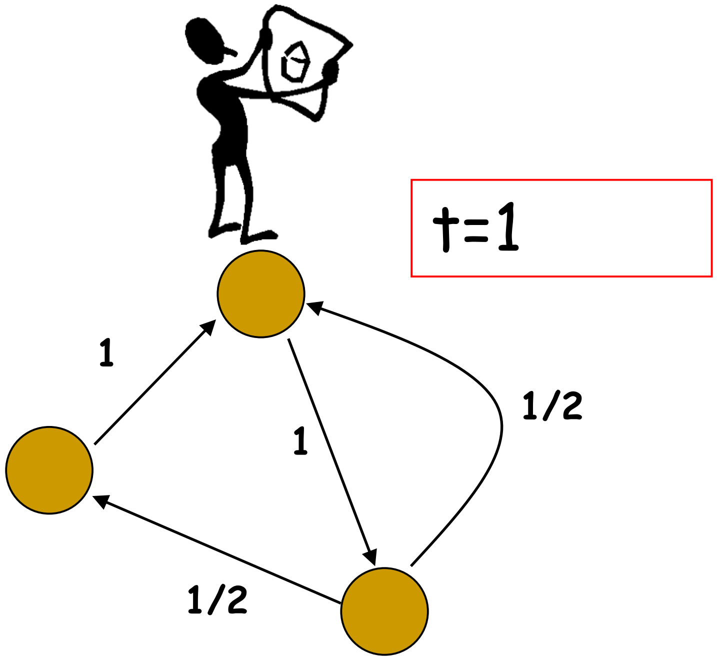


What is a random walk

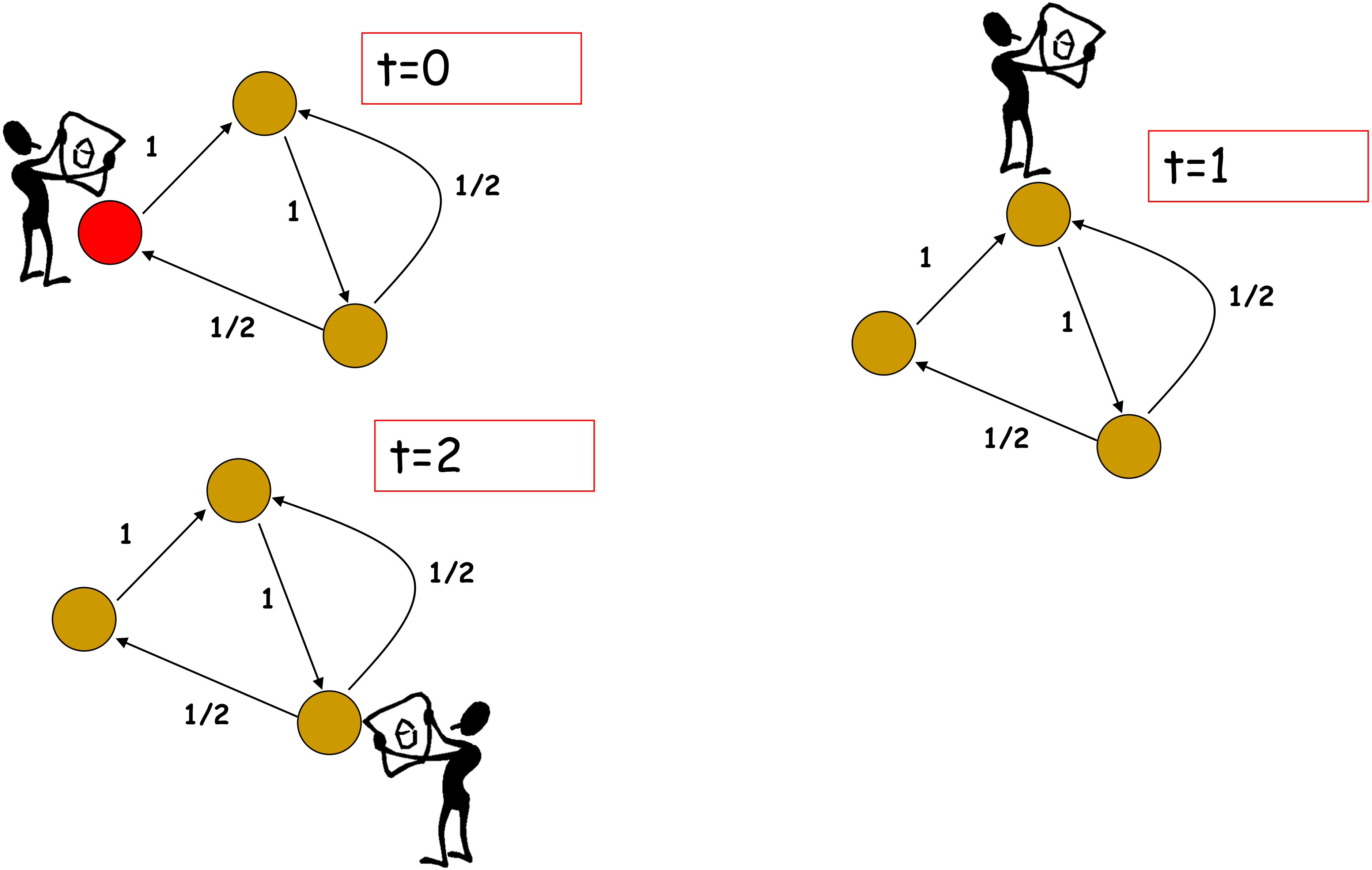


What is a random walk

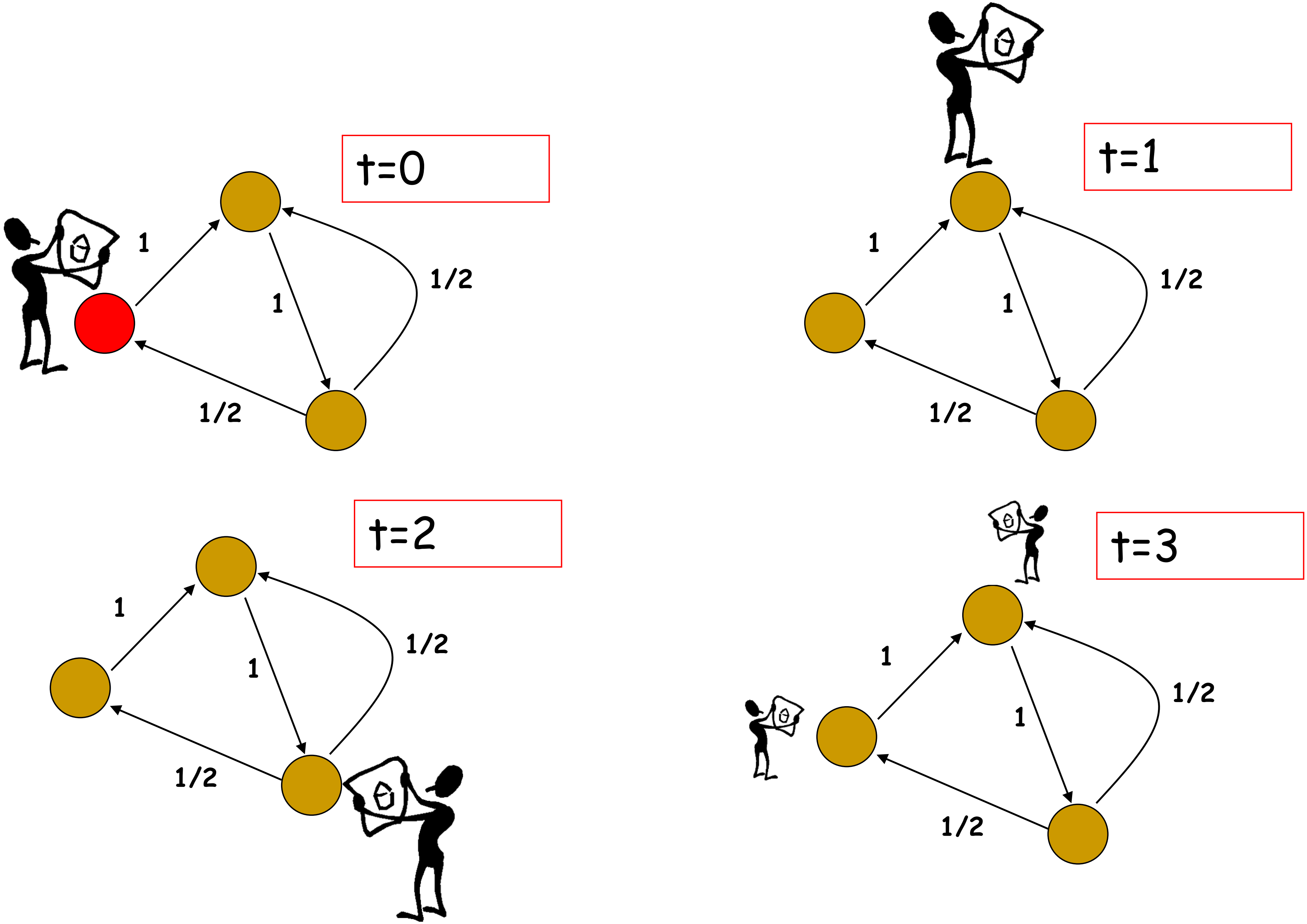




What is a random walk



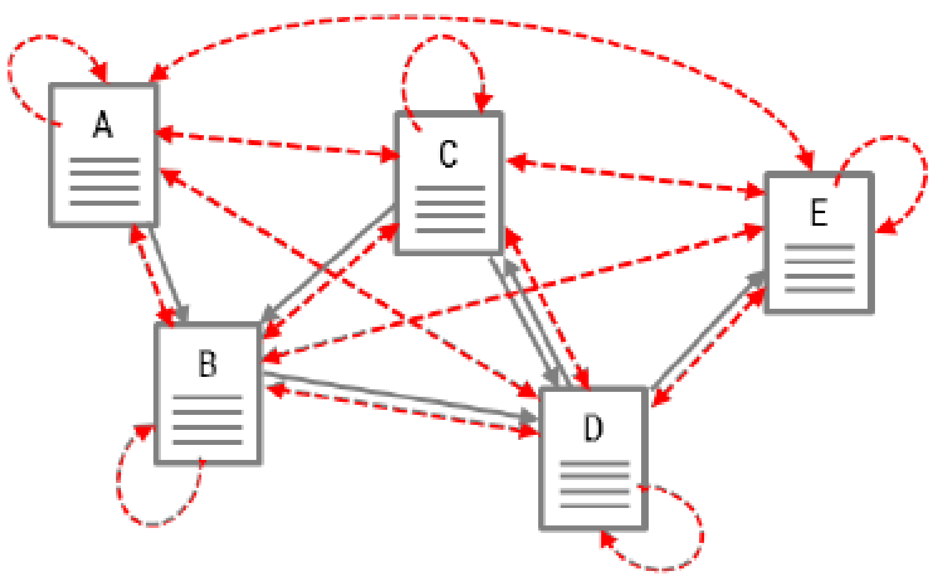
What is a random walk



Pagerank

"PageRank can be thought of as a model of user behavior. We assume there is a "random surfer" who is given a web page at random and keeps clicking on links, never hitting "back" but eventually gets bored and starts on another random page. The **probability** that the random surfer visits a page is its **PageRank**."

$$PR(A) = (1 - d) + d(PR(T1)/C(T1) + \ldots + PR(Tn)/C(Tn))$$



Pagerank

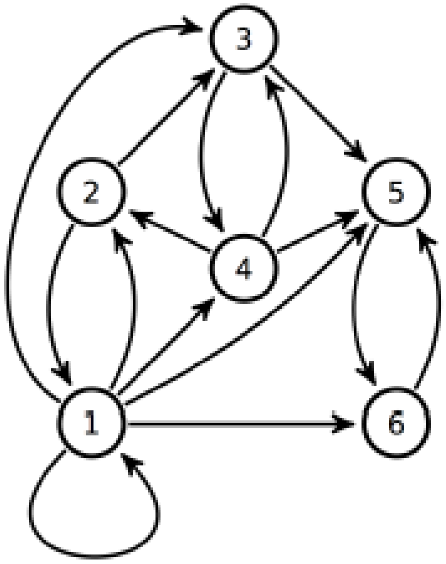
- Random walk on a directed graph

$$p_i^{t+1} = \sum_{j \in N(i)} \frac{p_j^t}{d_j^{out}} = \sum_j \frac{A_{ji}}{d_j^{out}} p_j$$

$$\mathbf{D}_{ii} = diag\{d_i^{out}\}$$

$$\mathbf{p}^{t+1} = (\mathbf{D}^{-1}\mathbf{A})^T \mathbf{p}^t$$

$$\mathbf{p}^{t+1} = \mathbf{P}^T \mathbf{p}^t$$



- Markov chain with transition probability matrix $\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}$

$$\lim_{t \to \infty} \mathbf{p}^t = \pi$$

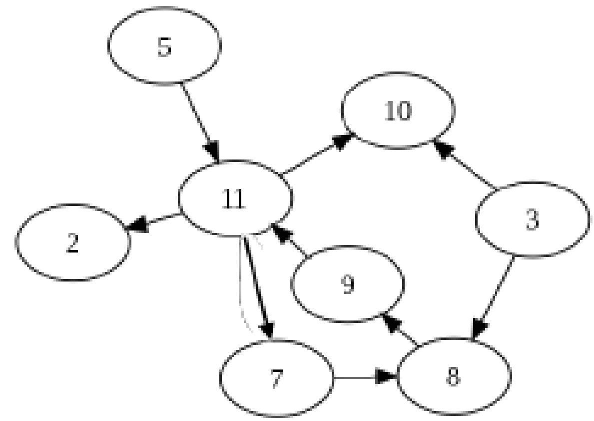
Interpreting web surfing

- Iinitially, every web page chosen uniformly at random
- With probability α, perform random walk on web by randomly choosing hyperlink in page ??
- With probability 1 - α, stop random walk and restart web surfing
- PageRank $\rightarrow$ steady state probability that a web page is visited through web surfing ??

Pagerank

Transition matrix:

$$\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}$$



Stochastic matrix:

$$\mathbf{P}' = \mathbf{P} + \frac{\mathbf{s}\mathbf{e}^T}{n}$$

PageRank matrix:

$$\mathbf{P}'' = \alpha\mathbf{P}' + (1-\alpha)\frac{\mathbf{e}\mathbf{e}^T}{n}$$

Eigenvalue problem (choose solution with $\lambda = 1$):

$$\mathbf{P}''^T\mathbf{p} = \lambda\mathbf{p}$$

Notations:

e - unit column vector, **s** - absorbing nodes indicator vector (column)

Outer product

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \\ u_4 v_1 & u_4 v_2 & u_4 v_3 \end{bmatrix}.$$

Perron-Frobenius Theorem

Perron-Frobenius theorem (Fundamental Theorem of Markov Chains)
If matrix is

- stochastic (non-negative and rows sum up to one, describes Markov chain)
- irreducible (strongly connected graph)
- aperiodic

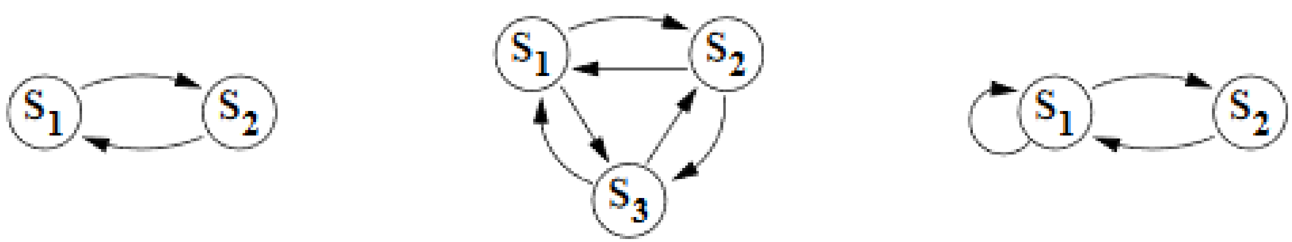


Figure 2.2: Examples of three Markov chains of which the left one has period 2 while the other two both are aperiodic.

Irreducible

Irreducible Markov chains

Let us consider Markov chains on a small state space $S = \{s_1, s_2, s_3, s_4, s_5\}$.

Some examples . . .

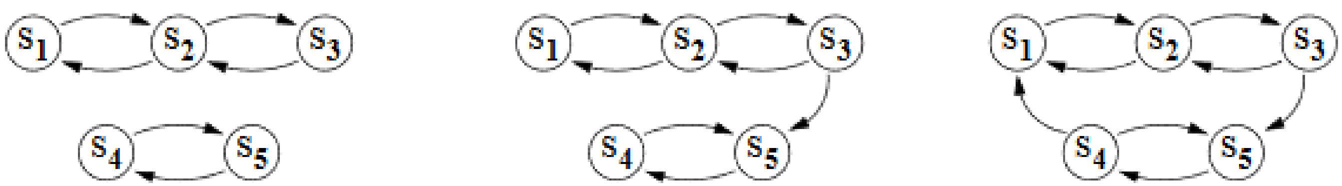


Figure 2.1: Examples of three Markov chains the one to the right is irreducible while the other two are not.

Irreducibility is the property that **regardless the present state we can reach any other state in finite time** . Mathematically it is expressed as . . .

Perron-Frobenius Theorem

Perron-Frobenius theorem (Fundamental Theorem of Markov Chains)
If matrix is

- stochastic (non-negative and rows sum up to one, describes Markov chain)
- irreducible (strongly connected graph)
- aperiodic

then

$$\exists \lim_{t\to\infty} \bar{\mathbf{p}}^t = \bar{\pi}$$

and can be found as a left eigenvector

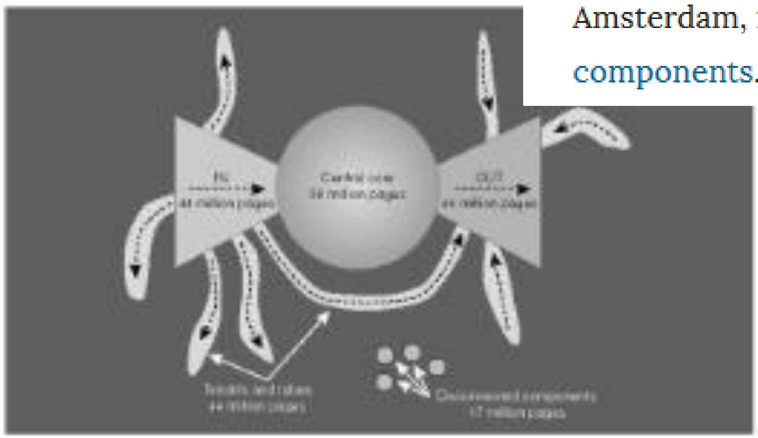
$$\bar{\pi}\mathbf{P} = \bar{\pi}, \quad \text{where} \quad ||\bar{\pi}||_1 = 1$$

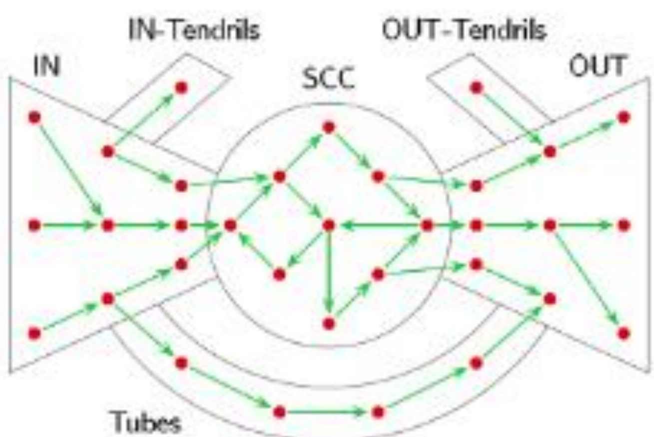
$\bar{\pi}$ - stationary distribution of Markov chain, raw vector

Bow tie structure of Web

Researchers from three Californian groups – at IBM's Almaden Research Center in San Jose, the Altavista search engine in San Mateo and Compaq Systems Research Center in Palo Alto – have analysed 200 million web pages and 1.5 billion hyperlinks. Their results, which will be presented next week at the World Wide Web 9 Conference in Amsterdam, indicate that the web is made up of four distinct components.

Bow tie structure of the web



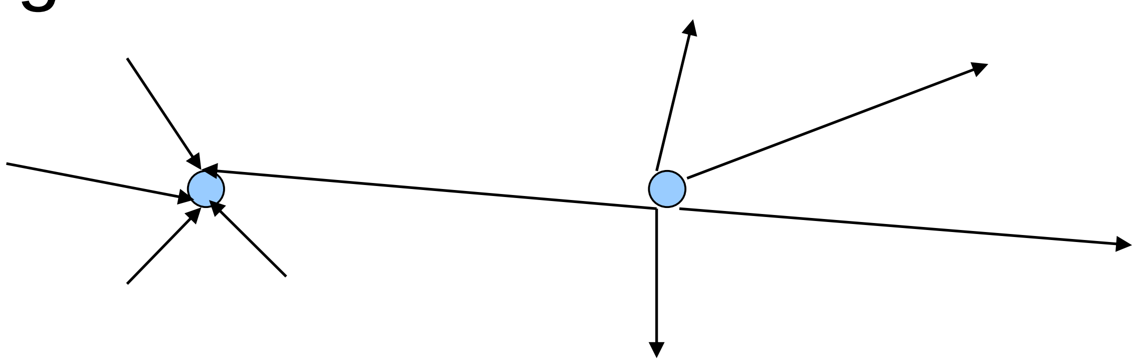


Govt. repository/pages, corporate pages

New pages

Hubs and Authorities

- Each node has two types of centralities: hub centrality, authority centrality
- authorities: nodes with useful (important) information (e.g., important scientific paper)
- hubs: nodes that tell where best authorities are (e.g., good review paper)
- Hyperlink-induced topic search (HITS) proposed by Kleinberg 1999 in J. ACM



Hubs and Authorities

Citation networks. Reviews vs original research (authoritative) papers

- authorities, contain useful information, a_i
- hubs, contains links to authorities, h_i

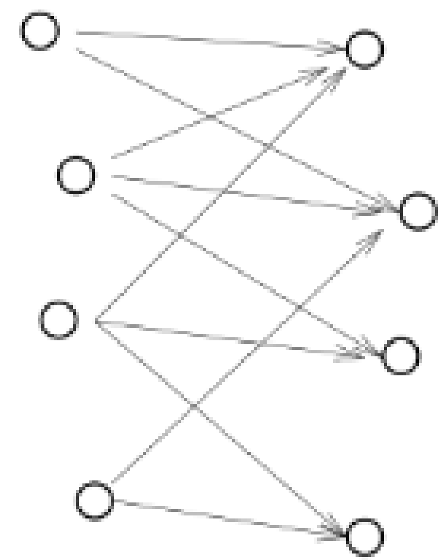
Mutual recursion

- Good authorities reffered by good hubs

$$a_i \leftarrow \sum_j A_{ji} h_j$$

- Good hubs point to good authorities

$$h_i \leftarrow \sum_j A_{ij} a_j$$



Hubs and Authorities

System of linear equations

$$\begin{aligned} \mathbf{a} &= \alpha \mathbf{A}^T \mathbf{h} \\ \mathbf{h} &= \beta \mathbf{A} \mathbf{a} \end{aligned}$$

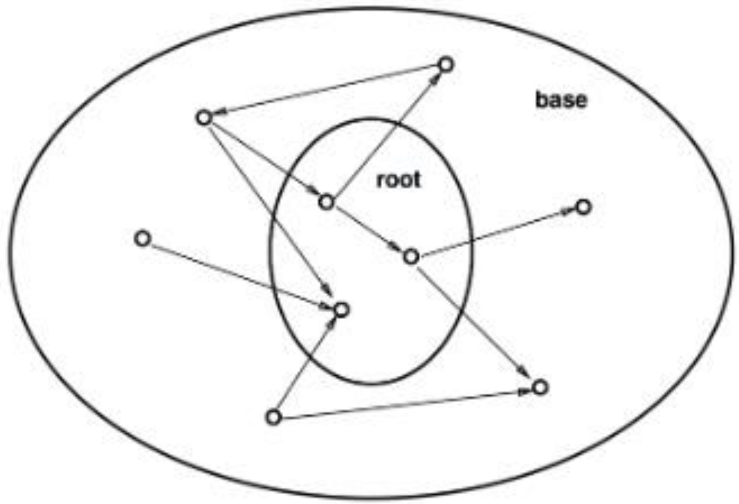
Symmetric eigenvalue problem

$$\begin{aligned} (\mathbf{A}^T \mathbf{A})\mathbf{a} &= \lambda \mathbf{a} \\ (\mathbf{A}\mathbf{A}^T)\mathbf{h} &= \lambda \mathbf{h} \end{aligned}$$

where eigenvalue $\lambda = (\alpha\beta)^{-1}$

HITS

Focused subgraph of WWW



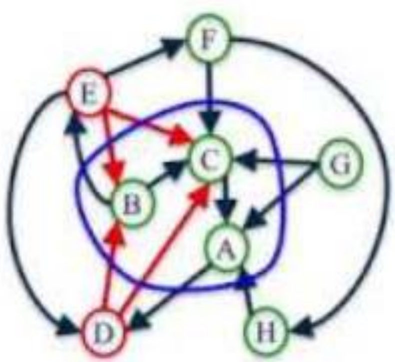
Root set
Base set

HITS Algorithm Convergence

For most networks, as k gets larger, authority and hub scores converge to a unique value.

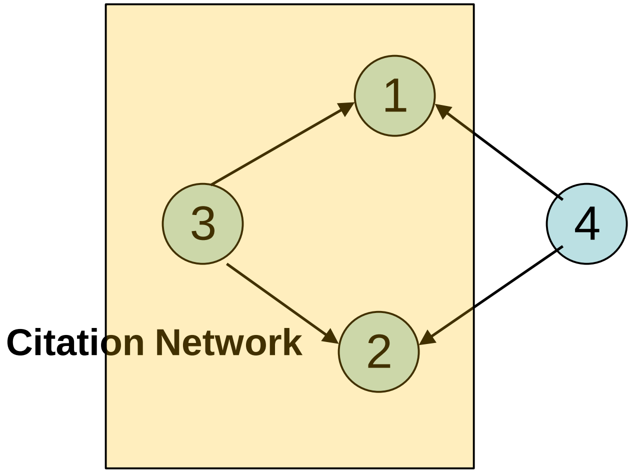
As $k \longrightarrow \infty$ the hub and authority scores approach:

	A	B	C	D	E	F	G	H
Auth	.08	.19	.40	.13	.06	.11	0	.06
Hub	.04	.14	.03	.19	.27	.14	.15	.03



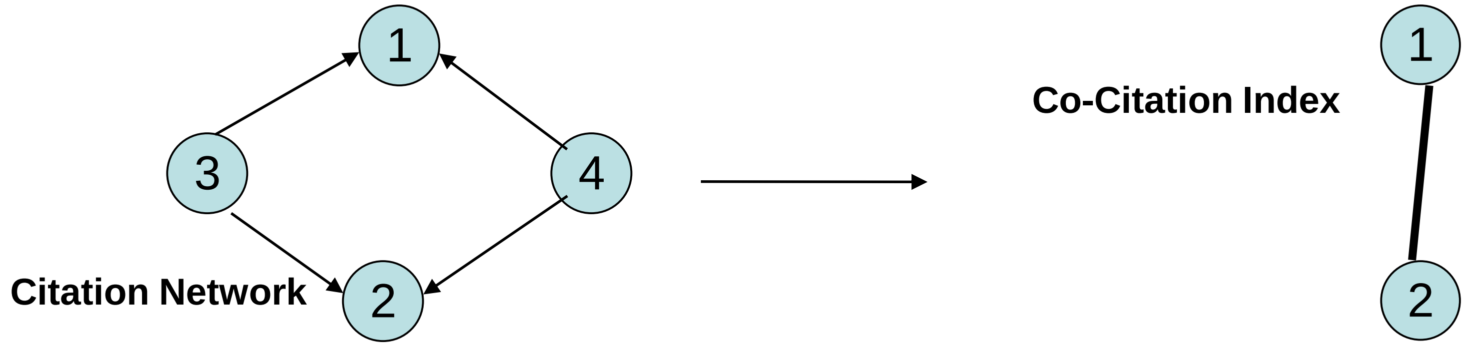
Co-citation Index

- Consider the following (co-citation)
  - Author 1 is cited by author 3
  - Author 2 is cited by author 3
- Either of 1 or 2 has never cited each other
- Can there be any relationship between author 1 and author 2??



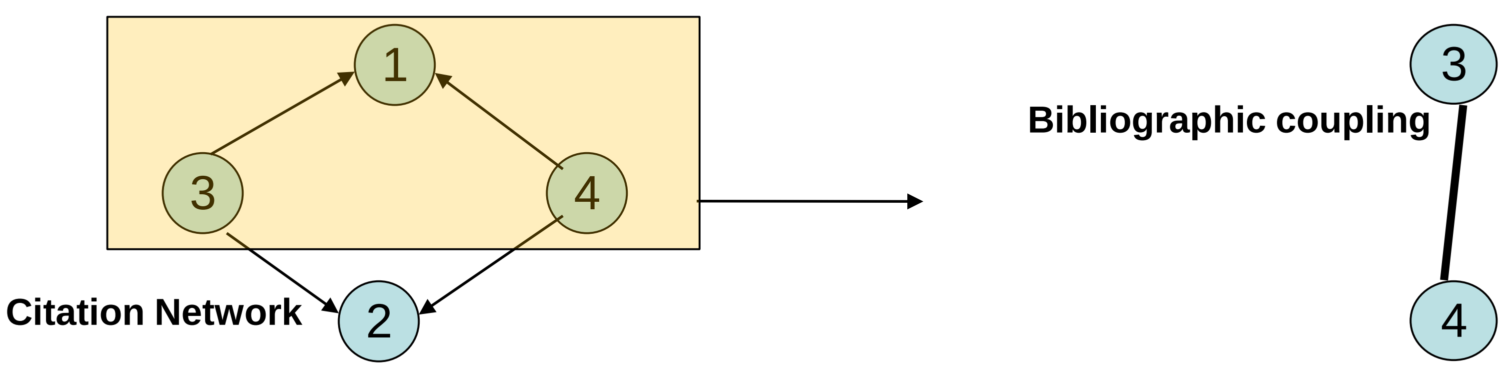
Co-citation Index

- Consider the following (co-citation)
  - Author 1 is cited by author 3
  - Author 2 is cited by author 3
- Either of 1 or 2 has never cited each other
- Can there be any relationship between author 1 and author 2?? Seems to be!! If you are not convinced consider that there are 1000 others like author 3
- There is a high chance that 1 and 2 work in related fields



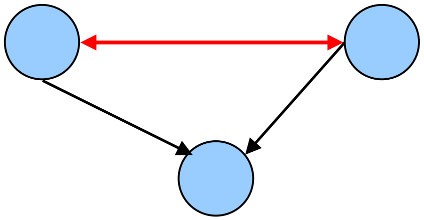
Bibliographic coupling

- Mirror Image: Consider the following
- Author 3 cites author 1
- Author 4 cites author 1
- Either of 3 or 4 has never cited each other
- Can there be any relationship between author 3 and author 4?? Agian it seems to be so!!
- 3 and 4 possibly works in the same field



Reciprocity

- If there is directed edge from node i to node j in directed network and there is also edge from node j to i, then edge from i to j is reciprocated.
- pairs of reciprocated edges called co-links.



- reciprocity r defined as fraction of edges that are reciprocated => $r = m^{-1}\sum_{ij} A_{ij}A_{ji}$

Rich-club Coefficient

- In science, influential researchers sometimes co-author a paper together (something strongly impactful)
- Hubs (usually high degree nodes) in a network are densely connected → A “rich club”
- The rich-club of degree k of a network $G = (V, E)$ is the set of vertices with degree greater than k, $R(k) = \{v \in V \mid k_v > k\}$. The rich-club coefficient of degree k is given by:

Directed

$$\frac{\#\mathrm{edge}(i,j)}{|R(k)||R(k)-1|}, \text{ where } (i,j) \in R(k)$$

Matching Index

- A *matching index* can be assigned to each edge in a network in order to quantify the similarity between the connectivity pattern of the two vertices adjacent to that edge
- Low value → Dis-similar regions of the network → a shortcut to distant regions
- Matching Index of edge(i,j):

$$\mu_{ij} = \frac{\sum_{k \neq i,j} A_{ik} A_{kj}}{\sum_{k \neq j} A_{ik} + \sum_{k \neq i} A_{jk}}$$