Tutorial 9
Complexity Theory

Time Complexity

1. For each of the following statements, answer *True*, *False* or *Open-Question* according to our current state of knowledge of complexity theory, as described in class. Give brief justifications for your answers.

   (a) $\mathbf{P} \subseteq \mathtt{TIME}\left(n^{2024}\right)$?

   (b) $\mathtt{SAT} \leq_P \overline{\mathtt{SAT}}$?

   (c) $\mathtt{HAMPATH} \leq_P \mathtt{PATH}$?

   (d) $\mathtt{PATH} \leq_P \overline{\mathtt{PATH}}$?

2. Prove that the following languages (defined over graphs) are in **P**.

   (a) BIPARTITE : the set of all bipartite graphs, i.e., $G = (V, E) \in$ BIPARTITE if V can be partitioned into two sets V_1, V_2 such that every edge in E is adjacent to a vertex in V_1 and a vertex in V_2 (no edge falls inside V_1 or V_2).

   (b) TRIANGLE-FREE : the set of all graphs that do not contain a triangle (where triangle is a set of three distinct vertices that are mutually connected).

3. Normally, we assume that numbers are represented as strings using the binary basis. That is, a number n is represented by the sequence $x_0, x_1, \ldots, x_{\log n}$ such that $n = \sum_{i=0}^{\log n} x^i 2^i$. However, we could have used other encoding schemes. If $n \in \mathbb{N}$ and $b \geq 2$, then the representation of n in base b, denoted by $\llcorner n \lrcorner_b$ is obtained as follows: first represent n as a sequence of digits in $\{0, \ldots, b-1\}$, and then replace each digit by a sequence of zeroes and ones. The unary representation of n, denoted by $\llcorner n \lrcorner_1$ is the string 1^n (that is, a sequence of n ones).

   (a) Show that choosing a different base of representation (other than unary) will make no difference to the class **P**. That is, show that for every subset S of the natural numbers, if we define $L_S^b = \left\{ \llcorner n \lrcorner_b \mid n \in S \right\}$, then for every $b \geq 2$, $L_S^b \in \mathbf{P}$ iff $L_S^2 \in \mathbf{P}$.

   (b) Show that choosing the unary representation makes a difference by showing that the following language is in **P**.

   $$\text{UNARY-FACTORING} = \left\{ \langle \llcorner n \lrcorner_1, \llcorner k \lrcorner_1 \rangle \mid \text{there is a } j \leq k \text{ dividing } n \right\}$$

4. Prove that **P** = **coP** and **P** ⊆ **NP** ∩ **coNP**.

5. Assuming **NP** ≠ **coNP**, show that no **NP-complete** problem can be in **coNP**.

6. Show that the halting problem is **NP-hard**.

7. Consider the following *solitaire* game. You are given an $m \times m$ board where each one of the m^2 positions may be empty or occupied by either a red stone or a blue stone. Initially, some configuration of stones is placed on the board. Then, for each column you must remove either all of the red stones in that column or all of the blue stones in that column. (If a column already has only red stones or only blue stones in it then you do not have to remove any further stones from that column.) The objective is to leave at least one stone in each row. Finding a *solution* that achieves this (mentioned) objective may or may not be possible depending upon the initial configuration. Let,

   $$\mathtt{SOLITAIRE} = \left\{ \langle G \rangle \mid G \text{ is a game configuration with a solution} \right\}$$

   Prove that, SOLITAIRE is **NP-complete**.

8. Let DOUBLE-SAT $= \{\langle\varphi\rangle \mid \varphi$ is a CNF formula having at least two satisfying assignments$\}$. Show that DOUBLE-SAT is **NP-complete**.

9. A *vertex cover* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that every edge of G is incident on at least one vertex in S. Show that the following language is **NP-complete**.

   $$\text{VERTEX-COVER} = \{(G, k) \mid \text{graph } G \text{ has a vertex cover of size} \leq k\}$$

10. Let S be a set and let $C = \{X_1, X_2, \ldots, X_n\}$ be a collection of n subsets of S (for each $i \in [1, n], X_i \subseteq S$). A set S', with $S' \subseteq S$, is called a *hitting set* for C if every subset in C contains at least one element in S', i.e., $|X_i \cap S'| \geq 1$ for each $i \in [1, n]$. Let HITSET $= \{\langle C, k\rangle \mid C$ has a hitting set of size $k\}$. Prove that HITSET is **NP-complete**.

    Example: $S = \{a, b, c, d, e, f, g\}$, $C = \{\{a, b, c\}, \{d, a\}, \{d, e, f\}, \{g\}\}$

    - $k = 2$, no hitting sets exist.
    - $k = 3$, $S' = \{a, d, g\}$ (other choices exist).

    Hint: Try reducing from VERTEX-COVER.

11. **[Scaling Resource Bounds]** Let $\mathbf{CL}_1, \mathbf{CL}_2$ denote some time/space complexity classes. Show that, if $\mathbf{CL}_1(f(n)) \subseteq \mathbf{CL}_2(g(n))$, then $\mathbf{CL}_1(f(n^c)) \subseteq \mathbf{CL}_2(g(n^c))$.

12. The following two classes are exponential time analogues of **P** and **NP**.

    $$\mathbf{EXP} = \bigcup_{c \geq 1} \mathbf{DTIME}(2^{n^c}) \qquad \text{and} \qquad \mathbf{NEXP} = \bigcup_{c \geq 1} \mathbf{NTIME}(2^{n^c})$$

    Clearly, $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$. Show that, if $\mathbf{EXP} \neq \mathbf{NEXP}$, then $\mathbf{P} \neq \mathbf{NP}$.

    Hint: Consider padding strings in **EXP**/**NEXP** *languages with exponentially sized strings in order to "scale down" to* **P**/**NP**.
