Tutorial 8
Computability Theory

Recursive and R.E. Languages, (Un)Decidability, Reduction and Rice's Theorem

1. For a language L over the alphabet $\{0, 1\}$, define the following two languages:

$$\mathsf{HALF}_1(L) = \{x \mid x \in \Sigma^*, \text{ and there exists } y \in \Sigma^* \text{ such that } |x| = |y| \text{ and } xy \in L\}$$
$$\mathsf{HALF}_2(L) = \{x \mid x \in \Sigma^*, \text{ and there exists } y \in \Sigma^* \text{ such that } |x| = |y| \text{ and } yx \in L\}$$

   Prove/Disprove the following (for $k = 1$ and $k = 2$):
   (a) If L is R.E. (recursively enumerable), then $\mathsf{HALF}_k(L)$ must be R.E..
   (b) If L is recursive, then $\mathsf{HALF}_k(L)$ must be recursive.

2. Consider the following languages for (finite) $k \in \mathbb{N}$:

$$\mathsf{LOOP}_{LE} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that loops on at most } k \text{ input strings}\}$$
$$\mathsf{LOOP}_{GE} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that loops on at least } k \text{ input strings}\}$$
$$\mathsf{LOOP}_{LT} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that loops on less than } k \text{ input strings}\}$$
$$\mathsf{LOOP}_{GT} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that loops on more than } k \text{ input strings}\}$$
$$\mathsf{LOOP}_{EQ} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that loops on exactly } k \text{ input strings}\}$$

   Determine whether the above languages (and the complements of these languages) are recursive, R.E., or non-R.E.

3. Consider the following languages for (finite) $k \in \mathbb{N}$:

$$\mathsf{HALT}_{LE} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that halts on at most } k \text{ input strings}\}$$
$$\mathsf{HALT}_{GE} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that halts on at least } k \text{ input strings}\}$$
$$\mathsf{HALT}_{LT} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that halts on less than } k \text{ input strings}\}$$
$$\mathsf{HALT}_{GT} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that halts on more than } k \text{ input strings}\}$$
$$\mathsf{HALT}_{EQ} = \{\mathcal{M} \mid \mathcal{M} \text{ is (the encoding of) a DTM that halts on exactly } k \text{ input strings}\}$$

   Determine whether the above languages (and the complements of these languages) are recursive, R.E., or non-R.E.

4. Let $\mathcal{N}$ be a Non-deterministic Turing machine (NTM). We say that $\mathcal{N}$ faces a dilemma if at some point in its working, it encounters a situation where the finite control is in the state p, the head scans the tape symbol a, and $\delta(p, a)$ offers multiple (two or more) possibilities, where p is neither the accept nor the reject state. Consider the following two languages.

$$\mathsf{DILEMMA}_\epsilon = \{\mathcal{N} \mid \mathcal{N} \text{ is an NTM which faces a dilemma at least once on input } \epsilon\},$$
$$\mathsf{DILEMMA}_\star = \{\mathcal{N} \mid \mathcal{N} \text{ is an NTM which faces a dilemma at least once on each input}\}.$$

   Answer the following.
   (a) Prove that $\mathsf{DILEMMA}_\epsilon$ is R.E., but not recursive.
   (b) Prove that $\mathsf{DILEMMA}_\star$ is non-R.E. (not recursively enumerable).

5. Show that the following language is decidable.

$$\text{ODFA} = \{\mathcal{M} \mid \mathcal{M} \text{ is a DFA not accepting any string with odd number of 1's}\}$$

(*Hint*: For a DFA $\mathcal{M}$, the problem of whether or not $\mathcal{L}(\mathcal{M}) = \phi$ is decidable.)

6. Recall that the halting problem for linear bounded automaton (LBA) is decidable. Prove by diagonalization that there exists a recursive set that is not accepted by any LBA.

7. Show, using reductions, that none of the following languages nor their complements are R.E.

   (a) $\text{REG} = \{\mathcal{M} \mid \mathcal{L}(\mathcal{M}) \text{ is a regular}\}$

   (b) $\text{CFL} = \{\mathcal{M} \mid \mathcal{L}(\mathcal{M}) \text{ is a context-free}\}$

   (c) $\text{REC} = \{\mathcal{M} \mid \mathcal{L}(\mathcal{M}) \text{ is a recursive}\}$

   (d) $\text{TOT} = \{\mathcal{M} \mid \mathcal{M} \text{ halts on all inputs}\}$

   Now, use Rice's theorems to prove the same (as asked above).

8. Let $f(x) = \begin{cases} 3x+1, & \text{if } x \text{ is odd} \\ x/2, & \text{if } x \text{ is even} \end{cases}$ for any natural number x. Define $\mathsf{C}(x)$ as the sequence $(x,\ f(x),\ f(f(x)),\ \ldots)$ which terminates if and when it hits 1. For example, if $x = 7$, then

$$\mathsf{C}(x) = (7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1)$$

   Computer tests have shown that $\mathsf{C}(x)$ hits 1 eventually for x ranging from 1 to $2^{68} \approx 2.95 \times 10^{20}$ (as of 2024). But, the question of whether $\mathsf{C}(x)$ ends at 1 for all $x \in \mathbb{N}$ is not proven. This is believed to be true and known as the *Collatz conjecture*. Suppose that MP were decidable by a TM $\mathcal{K}$. Use $\mathcal{K}$ to describe a TM that is guaranteed to prove or disprove Collatz conjecture.

9. Prove or Disprove: The language $\{\mathcal{M} \mid \mathcal{M} \text{ is a DTM that runs in time } O(n^3)\}$ is undecidable.

10. For a language $A \subseteq \Sigma^*$ (with $|\Sigma| \geq 2$), define $A^R = \{w^R \mid w \in A\}$, where w^R denotes the reverse of the string w. Is it decidable, for a given TM $\mathcal{M}$, whether $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{M})^R$?

11. Prove/Disprove whether the following problems on a TM $\mathcal{M}$ are decidable for (finite) $k \in \mathbb{N}$.

    (a) Decide whether $\mathcal{M}$ halts on some input within k steps.

    (b) Decide whether $\mathcal{M}$ halts on some input beyond k steps.

    (c) Decide whether $\mathcal{M}$ halts on all inputs within k steps.

    (d) Decide whether $\mathcal{M}$ halts on all inputs beyond k steps.

    (e) Decide whether $\mathcal{M}$ runs for at least k^k steps for input a^k.

    (f) Decide whether $\mathcal{M}$ runs for at most k^k steps for input a^k.

    (g) Decide whether $\mathcal{M}$ on input ϵ moves left at least k times.

    (h) Decide whether $\mathcal{M}$ on a given input w moves left at least k times.

12. Is the problem whether a TM on any input re-enters the start state decidable or not? Prove.

13. Prove that the following languages are not recursive.

    (a) $\text{TMB} = \{\mathcal{M} \# w \mid \mathcal{M} \text{ writes the blank symbol at some point of time on input } w\}$

    (b) $\text{TMS} = \{\mathcal{M} \# w \# \$ \mid \mathcal{M} \text{ writes the symbol } \$ \in \Gamma \text{ at some point of time on input } w\}$

14. Determine whether the following languages are recursive, R.E. or neither? Justify the answer.

    (a) $L_{GE} = \{\mathcal{M} \mid \mathcal{L}(\mathcal{M}) \text{ contains at least 2024 elements}\}$

    (b) $L_{LE} = \{\mathcal{M} \mid \mathcal{L}(\mathcal{M}) \text{ contains at most 2024 elements}\}$

    (c) $L_{EQ} = \{\mathcal{M} \mid \mathcal{L}(\mathcal{M}) \text{ contains exactly 2024 elements}\}$

15. Determine whether the following languages are recursive, R.E. or neither? Justify the answer.

    (a) $L_{AM} = \{\mathcal{M} \mid \mathcal{M}$ accepts at most 2024 input strings$\}$
    (b) $L_{AL} = \{\mathcal{M} \mid \mathcal{M}$ accepts at least 2024 input strings$\}$
    (c) $L_{AA} = \{\mathcal{M} \mid \mathcal{M}$ accepts all strings of length $\leq 2024\}$
    (d) $L_{AS} = \{\mathcal{M} \mid \mathcal{M}$ accepts some strings of length $\geq 2024\}$
    (e) $L_{NA} = \{\mathcal{M} \mid \mathcal{M}$ does not accept all strings of length $\geq 2024\}$
    (f) $L_{NS} = \{\mathcal{M} \mid \mathcal{M}$ does not accept some string of length $\leq 2024\}$

16. For two TMs $(\mathcal{M}, \mathcal{N})$, determine whether the following are *decidable*, *semi-decidable*, or *not*.

    (a) $\mathcal{M}$ takes more steps than $\mathcal{N}$ on input ϵ.
    (b) $\mathcal{M}$ does not take more steps than $\mathcal{N}$ on input ϵ.

17. Let $\texttt{nsteps}(\mathcal{M}, w)$ denote the number of steps taken by $\mathcal{M}$ on w. If $\mathcal{M}$ loops on w, take $\texttt{nsteps}(\mathcal{M}, w) = \infty$. If $\mathcal{N}$ also loops on v, take $\texttt{nsteps}(\mathcal{M}, w) = \texttt{nsteps}(\mathcal{N}, v)$. Prove whether the following languages are *Recursive*, or *R.E. but not Recursive*, or *non-R.E.*

    (a) $L_1 = \{\mathcal{M} \# \mathcal{N} \mid \texttt{nsteps}(\mathcal{M}, \epsilon) < \texttt{nsteps}(\mathcal{N}, \epsilon)\}$
    (b) $L_2 = \{\mathcal{M} \# \mathcal{N} \mid \texttt{nsteps}(\mathcal{M}, \epsilon) \leq \texttt{nsteps}(\mathcal{N}, \epsilon)\}$
    (c) $L_3 = \{\mathcal{M} \# \mathcal{N} \mid \texttt{nsteps}(\mathcal{M}, w) < \texttt{nsteps}(\mathcal{N}, v)$ for some $w, v\}$
    (d) $L_4 = \{\mathcal{M} \# \mathcal{N} \mid \texttt{nsteps}(\mathcal{M}, w) < \texttt{nsteps}(\mathcal{N}, v)$ for all $w, v\}$

18. Using reductions, prove that the following languages are not recursive (undecidable).

    (a) $L_a = \{\mathcal{M} \# \mathcal{N} \mid \mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{N})\}$
    (b) $L_b = \{\mathcal{M} \# \mathcal{N} \mid \mathcal{L}(\mathcal{M}) \subseteq \mathcal{L}(\mathcal{N})\}$
    (c) $L_c = \{\mathcal{M} \# \mathcal{N} \mid \mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N}) = \phi\}$
    (d) $L_d = \{\mathcal{M} \# \mathcal{N} \mid \mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N})$ is finite$\}$
    (e) $L_e = \{\mathcal{M} \# \mathcal{N} \mid \mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N})$ is regular$\}$
    (f) $L_f = \{\mathcal{M} \# \mathcal{N} \mid \mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N})$ is context-free$\}$
    (g) $L_g = \{\mathcal{M} \# \mathcal{N} \mid \mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N})$ is recursive$\}$
    (h) $L_h = \{\mathcal{M} \# \mathcal{N} \# \mathcal{P} \mid \mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N}) = \mathcal{L}(\mathcal{P})\}$

19. Prove/Disprove: *No non-trivial property of recursively enumerable languages is semidecidable.*

20. **[Generalization of Rice's Theorem for Pairs of R.E. Languages]** Consider the set of pairs of R.E. languages: $RE^2 = \{(L, L') \mid L, L' \in RE\}$. Answer the following.

    (a) Define a property of pairs of R.E. languages.
    (b) How do you specify a property of a pair of R.E. languages?
    (c) Which properties of pairs of R.E. languages should be called non-trivial?
    (d) Prove that every non-trivial property of pairs of R.E. languages is undecidable.

21. Use the previous exercise to prove that the following problems about pairs of R.E. languages are undecidable.

    (a) $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{N})$
    (b) $\mathcal{L}(\mathcal{M}) \subseteq \mathcal{L}(\mathcal{N})$
    (c) $\mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N}) = \phi$
    (d) $\mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N})$ is finite
    (e) $\mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N})$ is context-free
    (f) $\mathcal{L}(\mathcal{M}) \bigcap \mathcal{L}(\mathcal{N})$ is recursive
    (g) $\mathcal{L}(\mathcal{M}) \bigcup \mathcal{L}(\mathcal{N}) = \Sigma^*$
    (h) $\mathcal{L}(\mathcal{M}) \bigcup \mathcal{L}(\mathcal{N}) = \phi$
    (i) $\mathcal{L}(\mathcal{M}) \bigcup \mathcal{L}(\mathcal{N})$ is finite
    (j) $\mathcal{L}(\mathcal{M}) \bigcup \mathcal{L}(\mathcal{N})$ is recursive

22. Generalize Rice's theorem, Part 2 (monotone-property related), for pairs of R.E. languages.
