Tutorial 3
Discrete Mathematics

Countable and Uncountable Sets

1. Let A and B be uncountable sets with $A \subseteq B$. Prove or disprove: A and B are equinumerous.
2. Let A be an uncountable set and B a countably infinite subset of A. Prove or disprove: A is equinumerous with $A \setminus B$.
3. Prove that the real interval $[0, 1)$ is equinumerous with the unit square $[0, 1) \times [0, 1)$.
4. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Show that, $[a, b) \times [c, d)$ is equinumerous with $[0, 1)$.
5. Define a relation $\sim$ on $\mathbb{R}$ such that $a \sim b$ if and only if $a - b \in \mathbb{Q}$. Answer the following:
   (a) Prove that $\sim$ is an equivalence relation.
   (b) Is the set $\mathbb{R}/\sim$ of all equivalence classes of $\sim$ countable?
6. Let $f : S \longrightarrow \mathbb{N}$ be a one-one correspondence of set S with $\mathbb{N}$. Define a relation $\mathcal{R}_f$ on S as: $\mathcal{R}_f = \{(a, b) \in S^2 \mid f(a) \leq f(b)\}$. Prove that, $\mathcal{R}_f$ is a linear ordering on S such that every element of S has only finitely many predecessors under $\mathcal{R}_f$.
7. Let $\mathbb{Z}[x]$ denote the set of all univariate polynomials with integer coefficients.

   Answer the following:
   (a) Prove that $\mathbb{Z}[x]$ is countable.
   (b) A real or complex number a is called algebraic if $f(a) = 0$ for some non-zero $f(x) \in \mathbb{Z}[x]$. Let $\mathbb{A}$ denote the set of all algebraic numbers. Prove that $\mathbb{A}$ is countable.
   (c) Prove that there are uncountably many transcendental (i.e. non-algebraic) numbers.
8. Let $\mathbb{Z}[x, y]$ be the set of all bivariate polynomials with integer coefficients.

   Answer the following:
   (a) Prove that $\mathbb{Z}[x, y]$ is countable.
   (b) Let $V = \{(a, b) \in \mathbb{C} \times \mathbb{C} \mid f(a, b) = 0 \text{ for some nonzero } f(x, y) \in \mathbb{Z}[x, y]\}$. Is V countable?
9. A set $S \subseteq \mathbb{R}$ is called bounded if S has both a lower bound and an upper bound.

   Provide examples for the following.
   (a) Countable bounded subset of $\mathbb{R}$.
   (b) Uncountable bounded subset of $\mathbb{R}$.

   Determine whether the following sets are countable/uncountable?
   (c) The set of all bounded subsets of $\mathbb{Z}$.
   (d) The set of all bounded subsets of $\mathbb{Q}$.
10. Provide a diagonalization argument to prove that the set of all infinite bit sequences is uncountable.