Tutorial 2
Discrete Mathematics

Set, Relation and Function

1. Show the following.

   (a) If $a \in \{\{b\}\}$, then $b \in a$.

   (b) If $C = \{\{x\} \mid b \in B\}$, then $\bigcup_{X \in C} X = B$.

2. Prove that for all sets A, B the following statements are equivalent

   $$(a)\ A \subseteq B, \qquad (b)\ A \setminus B = \phi, \qquad (c)\ A \cup B = B, \qquad (d)\ A \cap B = A.$$

3. Let $A, B, C \in \mathcal{U}$ are three arbitrary sets such that, $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Prove that, $B = C$.

4. For two sets A, B, the symmetric difference $A \Delta B$ is defined as $(A \setminus B) \cup (B \setminus A)$. Prove the following for all sets A, B, C.

   (a) $A \Delta B = (A \cup B) \setminus (A \cap B)$

   (b) If $A \setminus C = B \setminus C$, then $A \Delta B \subseteq C$.

   (c) $A \Delta B = \phi$ if and only if $A = B$.

5. Let A be the set of all cities in India. Define a binary relation $\mathcal{R}$ on A as follows: for $x, y \in A$, $(x, y) \in \mathcal{R}$ if the distance between x and y is at most 2024 km. Determine whether or not $\mathcal{R}$ is reflexive, asymmetric, transitive, anti-symmetric or irreflexive.

6. A relation π is circular if $(x, y), (y, z) \in \pi$ implies $(z, x) \in \pi$. Prove that π is an equivalence relation if and only if it is both circular and reflexive.

7. Let ρ be a total order on A. We call ρ a *well-ordering* of A if every non-empty subset of A contains a least element. In this exercise, we plan to construct a well-ordering of $A = \mathbb{N} \times \mathbb{N}$.

   (a) Define a relation ρ on A as $(a, b)\ \rho\ (c, d)$ if and only if $a \leq c$ or $b \leq d$.

   (b) Define a relation σ on A as $(a, b)\ \sigma\ (c, d)$ if and only if $a \leq c$ and $b \leq d$.

   (c) Define a relation $\leq_L$ on A as $(a, b) \leq_L (c, d)$ if either (i) $a < c$, or (ii) $a = c$ and $b \leq d$.

   Prove or disprove: ρ, σ, $\leq_L$ is a well-ordering of A.

8. Let A be the set of all functions $\mathbb{N}_0 \to \mathbb{R}^+$. Define a relation Θ on A as $f\ \Theta\ g$ if and only if $f = \Theta(g)$. Define a relation O on A as $f\ O\ g$ if and only if $f = O(g)$. Answer the following:

   (a) Prove that Θ is an equivalence relation.

   (b) Argue that O is not a partial order.

   Redefine the relation O on A/Θ as $[f]\ O\ [g]$ if and only if $f = O(g)$. Answer the following:

   (c) Establish that the relation O is well-defined.

   (d) Prove that O is a partial order on A/Θ.

   (e) Prove or disprove: O is a total order on A/Θ.

   (f) Prove or disprove: A/Θ is a lattice under O.

9. **[Genesis of rational numbers]** Define a relation ρ on $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $(a, b)\ \rho\ (c, d)$ if and only if $ad = bc$. Prove that ρ is an equivalence relation. Argue that A/ρ is essentially the set $\mathbb{Q}$ of rational numbers.

10. Which of the following are bijections? Justify your answer.
    (a) $f : \mathbb{Z} \to \mathbb{Z}$, where $f(n) = (-1)^{|n|} n$ for every $n \in \mathbb{Z}$.
    (b) $f : \mathbb{Z} \to \mathbb{Z}_{19}$, where $f(n) = n\ mod\ 19$ for every $n \in \mathbb{Z}$.
    (c) $f : \mathbb{R} \to C$, where $f(x) = x^3$ for all $x \in \mathbb{R}$.
    (d) $f : \mathbb{N} \to \mathbb{N}$, where $f(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$.

11. For a function, $f : A \to B$, define a function $\mathcal{F} : \mathcal{P}(A) \to \mathcal{P}(B)$ as $\mathcal{F}(S) = f(S)$ for all $S \subseteq A$. Here, $f(S) = \{f(s) \mid s \in S\}$. Prove the following:
    (a) $\mathcal{F}$ is injective if and only if f is injective.
    (b) $\mathcal{F}$ is surjective if and only if f is surjective.

12. Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a\ \rho\ a'$ if and only if $f(a)\ \sigma\ f(a')$. Answer the following:
    (a) Prove that, ρ is an equivalence relation on A.
    (b) Prove or disprove: ρ defines a partial order over A.

    Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$. Answer the following:
    (c) Prove that, $\bar{f}$ is well-defined.
    (d) Prove that, $\bar{f}$ is injective.
    (e) Prove or disprove: If f is a bijection, then so also is $\bar{f}$.
    (f) Prove or disprove: If $\bar{f}$ is a bijection, then so also is f.

13. Let S be a set. A characteristic function over S is a function $\chi : S \to \{0, 1\}$. If $A \subseteq S$, then the characteristic function of A is $\chi_A : S \to \{0, 1\}$ given by $\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$. Prove that there exists a bijection between 2^S and the set of all Boolean function on S, that is, $(S \to \{0, 1\})$.

14. Let Σ be an alphabet which is totally ordered, that is, for every $a, b \in \Sigma$, either $a \le b$ or $b \le a$. Consider the set Σ^*. We define the lexicographic ordering $\le$ on Σ^* as follows. Let $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_m$ be two strings in Σ^* with $\{x_i\}_i, \{y_j\}_j \in \Sigma$. We say $x \le y$ if: $n < m$ or $n = m$ and there exists an index $i \in \{1, 2, \ldots, n\}$ such that $x_j = y_j$ for all $1 \le j \le i-1$ and $x_i \le y_i$. Answer the following.
    (a) Show that $(\Sigma^*, \le)$ is a partial order. Is it a total order?
    (b) Does there exist a least element? What is it?
    (c) What is the greatest element?
    (d) Consider the subset $A = \{x \in \Sigma^* \mid l_1 \le |x| \le l_2\}$, where $l_1, l_2 \in \mathbb{Z}^+$ with $l_1 \le l_2$. What are the minimal/maximal elements of A? Are there least/greatest elements?
    (e) For the set A defined above, write down two different upper bounds and lower bounds. Is there a least upper bound and a greatest lower bound? What are they?

15. Prove the following.
    (a) If we inverse the relation in a partial order, then also it remains a partial order.
    (b) All finite lattices are complete.
    (c) Any sub-lattice of a complete lattice is also complete.