Tutorial 10
Complexity Theory

Space Complexity

1. For each of the following statements, answer *True*, *False* or *Open-Question* according to our current state of knowledge of complexity theory, as described in class. Give brief justifications for your answers.
   (a) **NSPACE**$(n^{2024}) \subseteq$ **PSPACE** ?
   (b) For any languages A, B and C, if $A \leq_L B$ and $B \leq_L C$ then $A \leq_L C$?
   (c) TQBF $\leq_L$ PATH ?
   (d) PATH $\leq_L \overline{\text{PATH}}$?

2. Suppose ODD-PARITY $= \{w \mid w \in \{0,1\}^* \text{ and } w \text{ contains an odd number of 1s}\}$. Recall that PATH $= \{\langle G, a, b\rangle \mid$ there is a path from a to b in directed graph $G\}$. Answer the following.
   (a) Prove: There is a log-space reduction from ODD-PARITY to PATH (ODD-PARITY $\leq_L$ PATH).
   (b) No log-space reduction from PATH to ODD-PARITY is known. State what surprising consequence would follow if such a reduction were discovered, and why.

3. Show that,
   (a) **PSPACE** is closed under union, intersection, complement and Kleene star operations.
   (b) **NL** is closed under union, intersection and Kleene star operations.

4. In the generalized version of the game Tic-Tac-Toe, 2 players places marks X (crosses) and O (noughts) on an $m \times n$ grid. A player wins if she is the first to place k marks in a row, column or diagonal. The game ends in a draw if no such sequence is present when all the mn cells of the grid are filled. Assuming that X always starts, show that the following language,

   $$\text{GTICTACTOE} = \{\langle m, n, k, c\rangle \mid c \text{ is an intermediate configuration on the } m \times n \text{ board with next move by } X \text{ and } \exists \text{ a winning strategy for } X\},$$

   is in **PSPACE**.

5. Let **polyL** $= \bigcup_{c>0}$ **SPACE**$(\log^c n)$. Let **SC** (named after Stephen-Cook) be the class of languages that can be decided by deterministic machines that run in polynomial time and $\log^c n$ space for some $c > 0$.
   (a) It is an open problem whether PATH $\in$ **SC**, where

   $$\text{PATH} = \{\langle G, s, t\rangle \mid \text{there exists a path from } s \text{ to } t \text{ in graph } G\}$$

   Why does Savitch's theorem not resolve this question? Explain.
   (b) Is **SC** $=$ **polyL** $\cap$ **P**? Justify.

6. Show that, 2SAT is in **NL**. Further, show that 2SAT is in **NL-complete**.

7. Define CYCLE $= \{\langle G\rangle \mid G \text{ is a directed graph that contains a directed cycle}\}$.
   Show that CYCLE is **NL-complete**.

8. Define UCYCLE $= \{\langle G\rangle \mid G \text{ is an undirected graph that contains a simple cycle}\}$.
   Show that UCYCLE $\in$ **L**. (Note: G may be a graph that is not connected.)