	INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR	
		Stamp / Signature of the Invigilator

EXAMINATION (Mid Semester)	SEMESTER (Autumn 2024-2025)

Roll Number									Section		Name	
Subject Number	C	S	6	0	0	5	Subject Name	*FOUNDATIONS OF COMPUTING SCIENCE*				
Department / Center of the Student							Additional sheets					

Important Instructions and Guidelines for Students

1. You must occupy your seat as per the Examination Schedule/Sitting Plan.
2. Do not keep mobile phones or any similar electronic gadgets with you even in the switched off mode.
3. Loose papers, class notes, books or any such materials must not be in your possession, even if they are irrelevant to the subject you are taking examination.
4. Data book, codes, graph papers, relevant standard tables/charts or any other materials are allowed only when instructed by the paper-setter.
5. Use of instrument box, pencil box and non-programmable calculator is allowed during the examination. However, exchange of these items or any other papers (including question papers) is not permitted.
6. Write on both sides of the answer script and do not tear off any page. **Use last page(s) of the answer script for rough work.** Report to the invigilator if the answer script has torn or distorted page(s).
7. It is your responsibility to ensure that you have signed the Attendance Sheet. Keep your Admit Card/Identity Card on the desk for checking by the invigilator.
8. You may leave the examination hall for wash room or for drinking water for a very short period. Record your absence from the Examination Hall in the register provided. Smoking and the consumption of any kind of beverages are strictly prohibited inside the Examination Hall.
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10. During the examination, either inside or outside the Examination Hall, gathering information from any kind of sources or exchanging information with others or any such attempt will be treated as **'unfair means'**. Do not adopt unfair means and do not indulge in unseemly behavior.

Violation of any of the above instructions may lead to severe punishment.

Signature of the Student

To be filled in by the examiner

Question Number	1	2	3	4	5	6	7	8	9	10	Total
Marks Obtained											

Marks obtained (in words)	Signature of the Examiner	Signature of the Scrutineer

Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

Foundation of Computing Science (CS60005) | **Autumn Semester 2024-2025**

19-September-2024 | **Mid-Semester Examination** | Maximum Marks: 60

Instructions:

- Write your answers in the question paper itself. Be brief and precise. Answer *all* questions.
- Write the answers only in the respective spaces provided. The last two blank pages may be used for rough work or leftover answers.
- In case you may need more space/pages, please ask for additional sheets in the exam hall and attach the same with this booklet while submitting.
- If you use any theorem / result / formula covered in the class, just mention it, do not elaborate. (unless the same thing has been explicitly asked to derive / prove in the question)
- Write all the proofs in mathematically / logically precise language. Unclear and/or dubious statements would be severely penalized.

— Question Paper Starts from Next Page —

Q1. Encode the five English statements given below, into well-formed predicate-logic formulas. Your encodings should use ***only*** the following predicates with the given meanings. **(2 × 5)**

$\text{boy}(x)$	:	x is a boy
$\text{girl}(x)$	:	x is a girl
$\text{love}(x,y)$	:	x loves y
$\text{marry}(x,y)$	:	x marries y
$\text{diff}(x,y)$	:	x and y are different

(a) Every boy who loves a girl does not love every other boy whom that girl loves.
Solution:

$$\forall x \forall y \left[\Big(\text{boy}(x) \wedge \text{girl}(y) \wedge \text{love}(x,y) \Big) \Rightarrow \forall z \left[\Big(\text{boy}(z) \wedge \text{diff}(x,z) \wedge \text{love}(y,z) \Big) \Rightarrow \neg \text{love}(x,z) \right] \right]$$

(b) Every boy who loves a girl marries that girl, but not every girl who marries a boy loves that boy.
Solution:

$$\Big(\forall x \forall y \left[\Big(\text{boy}(x) \wedge \text{girl}(y) \wedge \text{love}(x,y) \Big) \Rightarrow \text{marry}(x,y) \right] \Big)$$
$$\bigwedge \Big(\exists x \exists y \left[\text{boy}(x) \wedge \text{girl}(y) \wedge \text{marry}(y,x) \wedge \neg \text{love}(y,x) \right] \Big)$$

(c) Everyone loves either himself/herself or everyone else except himself/herself.

Solution:

$$\forall x \left[\text{love}(x,x) \bigvee \forall y \left(\text{diff}(x,y) \Rightarrow \text{love}(x,y) \right) \right]$$

(d) Every girl loves exactly one boy.

Solution:

$$\forall x \left[\text{girl}(x) \Rightarrow \exists y \left(\text{boy}(y) \wedge \text{love}(x,y) \bigwedge \forall z \left[\left(\text{boy}(z) \wedge \text{diff}(y,z) \right) \Rightarrow \neg \text{love}(x,z) \right] \right) \right]$$

(e) Every boy loves at least two girls.

Solution:

$$\forall x \left[\text{boy}(x) \Rightarrow \exists y \exists z \left(\text{girl}(y) \wedge \text{girl}(z) \wedge \text{love}(x,y) \wedge \text{love}(x,z) \wedge \text{diff}(y,z) \right) \right]$$

Q2. Suppose, ρ and σ are two binary relations defined over the set $\mathscr{A}$. A *composite relation* $\rho \circ \sigma$ over $\mathscr{A}$ is defined as, $\rho \circ \sigma = \{(p,r) \mid \text{ there exists some } q \in \mathscr{A}, \text{ such that } (p,q) \in \rho \text{ and } (q,r) \in \sigma\}$.

Prove the following assertions with precise formal justifications.

(a) If ρ and σ are two equivalence relations over $\mathscr{A}$, then prove that $\rho \circ \sigma$ is an equivalence relation *if and only if* $\rho \circ \sigma = \sigma \circ \rho$. **(7)**

Solution:

[$\Rightarrow$]

Suppose that $(x,y) \in \rho \circ \sigma$ $(x,y \in \mathscr{A})$. Since $\rho \circ \sigma$ is an equivalence relation, we also have $(y,x) \in \rho \circ \sigma$ (symmetric property). This means that for some $\alpha \in \mathscr{A}$, we have $(y,\alpha) \in \rho$ and $(\alpha,x) \in \sigma$. Since ρ and σ are both equivalence relations, we further get $(\alpha,y) \in \rho$ and $(x,\alpha) \in \sigma$ (symmetric property). This means that $(x,y) \in \sigma \circ \rho$ (by definition). Therefore $\rho \circ \sigma \subseteq \sigma \circ \rho$. Similar arguments (in opposite direction) can be given to prove $\sigma \circ \rho \subseteq \rho \circ \sigma$, thereby establishing $\rho \circ \sigma = \sigma \circ \rho$. [2 marks]

[$\Leftarrow$]

- Since ρ and σ are both equivalence relation, $(x,x) \in \rho$ as well as $(x,x) \in \sigma$ (for all $x \in \mathscr{A}$). By the definition of composite relations, we immediately have $(x,x) \in \rho \circ \sigma$, proving that $\rho \circ \sigma$ is *reflexive*. [1 mark]

- If $(x,y) \in \rho \circ \sigma$ $(x,y \in \mathscr{A})$, then for some $\alpha \in \mathscr{A}$, we have $(x,\alpha) \in \rho$ and $(\alpha,y) \in \sigma$. Since ρ and σ are both equivalence relations, we have $(\alpha,x) \in \rho$ and $(y,\alpha) \in \sigma$ (symmetric property). This means that $(y,x) \in \sigma \circ \rho$ (by definition). Finally, $\rho \circ \sigma = \sigma \circ \rho$ implies $(y,x) \in \rho \circ \sigma$. This proves that $\rho \circ \sigma$ is *symmetric*. [2 marks]

- Let $x,y,z \in \mathscr{A}$. Suppose that $(x,y) \in \rho \circ \sigma$ and $(y,z) \in \rho \circ \sigma$. Since $(x,y) \in \rho \circ \sigma$, there exists $\alpha \in \mathscr{A}$ such that $(x,\alpha) \in \rho$ and $(\alpha,y) \in \sigma$. Since $(y,z) \in \rho \circ \sigma$, there exists $\beta \in \mathscr{A}$ such that $(y,\beta) \in \rho$ and $(\beta,z) \in \sigma$. But then, since $(\alpha,y) \in \sigma$ and $(y,\beta) \in \rho$, we have $(\alpha,\beta) \in \sigma \circ \rho$ (by definition). It is given that $\sigma \circ \rho = \rho \circ \sigma$, so $(\alpha,\beta) \in \rho \circ \sigma$, that is, there exist $\delta \in \mathscr{A}$, such that $(\alpha,\delta) \in \rho$, and $(\delta,\beta) \in \sigma$. Since ρ is transitive, and (x,α) and (α,δ) are in ρ, we have $(x,\delta) \in \rho$. Moreover, since σ is transitive, and (δ,β) and (β,z) are in σ, we have $(\delta,z) \in \sigma$. By definition, we then have $(x,z) \in \rho \circ \sigma$, that is, $\rho \circ \sigma$ is *transitive*. [2 marks]

(b) The *inverse* of a relation τ over $\mathscr{A}$ is defined as, $\tau^{-1} = \{(q,p) \mid (p,q) \in \tau \text{ and } p,q \in \mathscr{A}\}$. Prove that, $(\rho \circ \sigma)^{-1} = (\sigma^{-1} \circ \rho^{-1})$. **(4)**

Solution:

Let $(y,x) \in (\rho \circ \sigma)^{-1}$ for $x,y \in \mathscr{A}$. By definition, $(x,y) \in (\rho \circ \sigma)$, that is, for some $\alpha \in \mathscr{A}$, we have $(x,\alpha) \in \rho$ and $(\alpha,y) \in \sigma$. This also implies that $(\alpha,x) \in \rho^{-1}$ and $(y,\alpha) \in \sigma^{-1}$. Therefore, $(y,x) \in \sigma^{-1} \circ \rho^{-1}$, concluding $(\rho \circ \sigma)^{-1} \subseteq (\sigma^{-1} \circ \rho^{-1})$.

On the other hand, let $(y,x) \in \sigma^{-1} \circ \rho^{-1}$ for $x,y \in \mathscr{A}$. Then, for some $\alpha \in \mathscr{A}$, we have $(y,\alpha) \in \sigma^{-1}$ and $(\alpha,x) \in \rho^{-1}$ (by definition). This also implies that $(\alpha,y) \in \sigma$ and $(x,\alpha) \in \rho$. Therefore, $(x,y) \in \rho \circ \sigma$ implies $(y,x) \in (\rho \circ \sigma)^{-1}$, concluding that $(\sigma^{-1} \circ \rho^{-1}) \subseteq (\rho \circ \sigma)^{-1}$.

Together, we prove that, $(\rho \circ \sigma)^{-1} = (\sigma^{-1} \circ \rho^{-1})$.

Q3. **(a)** Let $(G, \circ)$ be a group and $c \in G$. Define a binary composition $*$ on G by $a * b = a \circ c \circ b$ for all $a, b \in G$. Show that $(G, *)$ is a group, clearly indicating all the properties of a group. **(4)**

Solution:

$(G, *)$ is a group, because it satisfies the following properties of a group.

Closure: For any $p, q \in G$, $p * q = p \circ c \circ q \in G$, as $c \in G$ and G is closed under the operation $\circ$.

Associativity: For any $p, q, r \in G$, since G is associative under the operation $\circ$, we get:

$$(p * q) * r = (p \circ c \circ q) \circ c \circ r = p \circ c \circ (q \circ c \circ r) = p * (q * r)$$

Identity: c^{-1} is the identity element. For any element $p \in G$, we get:

$$p * c^{-1} = p \circ c \circ c^{-1} = p \circ e_G = p \quad \text{and} \quad c^{-1} * p = c^{-1} \circ c \circ p = e_G \circ p = p$$

where, $e_G \in G$ is the identity element with respect to the group $(G, \circ)$.

Inverse: For any element $p \in G$, let $p' \in G$ be its inverse. Now, by definition we should get $p * p' = c^{-1} = p' * p$.

$$\therefore\ p \circ c \circ p' = c^{-1} \quad \text{or} \quad p' \circ c \circ p = c^{-1} \quad \Longrightarrow \quad p' = c^{-1} \circ p^{-1} \circ c^{-1}$$

where, p^{-1} is the inverse of p with respect to the operation $\circ$.

(b) Let G be a multiplicative group. Prove that, if $(ab)^2 = a^2b^2$ for all $a, b \in G$, then G is abelian. **(3)**

Solution:

Take any two elements $a, b \in G$. Since $(ab)^2 = a^2b^2$, we have

$$e = (ab)^{-1}(ab)^2(ab)^{-1} = (b^{-1}a^{-1})(a^2b^2)(b^{-1}a^{-1}) = b^{-1}aba^{-1}.$$

This in turn implies that

$$ba = bea = b(b^{-1}aba^{-1})a = (bb^{-1})ab(a^{-1}a) = ab.$$

Q4. **(a)** Suppose that L_1 and L_2 are two languages (over the same alphabet) given to you such that L_1 and L_1L_2 (concatenation) are both regular. Prove or disprove: L_2 must be regular too. **(3)**

Solution:

This is *false*.

For example, take

$$L_1 = \mathscr{L}(a^*) \quad \text{and} \quad L_2 = \left\{a^p \mid p \text{ is a prime}\right\}.$$

But then,

$$L_1L_2 = \left\{a^n \mid n > 2\right\} = \mathscr{L}(aaa^*).$$

(b) Consider the language, $L_3 = \{a^i b^j \mid i, j \geq 0 \text{ and } |i-j| \text{ is a prime}\}$. (Note that, 1 is not a prime.) Using the pumping lemma for regular languages, prove that the language L_3 is *not* regular. **(4)**

Solution:

Suppose that L_3 is regular. Let k be a pumping-lemma constant for L_3.

Feed the string $\alpha\beta\gamma = a^{k+2}b^k$ with $\alpha = \varepsilon$, $\beta = a^{k+2}$ and $\gamma = b^k$, to the pumping lemma. We get a decomposition $\beta = \beta_1\beta_2\beta_3$ with $1 \leq l = |\beta_2| \leq k$.

Now, take $i = 3$, that is, pump in β_2 twice in $\alpha\beta\gamma$ to get the string $\alpha\beta_1\beta_2^3\beta_3\gamma = a^{k+2+2l}b^k \in L_3$. This is a contradiction, since $2 + 2l = 2(1+l)$ is not a prime.

(c) Consider the language, $L_4 = \{\alpha \in \{a,b\}^* \mid |\alpha| = n^2 \text{ for some integer } n \geq 0\}$, where $|\alpha|$ denotes the length of the string α. Using the pumping lemma for context-free languages, prove that the language L_4 is *not* context-free. **(4)**

Solution:

Assume that L_4 is context-free, and let n be a pumping lemma (PL) constant for L_4. Take $\alpha = a^{n^2}$. The PL gives us a decomposition $\alpha = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ with $0 < |\alpha_2\alpha_4| = k \leq n$ and with $\alpha' = \alpha_1\alpha_2^2\alpha_3\alpha_4^2\alpha_5 \in L_4$. But $n^2 < |\alpha'| = n^2 + k < (n+1)^2$, a contradiction!

Q5. Consider the language, $L_5 = \{\alpha \in \{0,1,2\}^* \mid \alpha \text{ does not contain two consecutive 0's}\}$.

(a) Describe a regular grammar for L_5. **(4)**

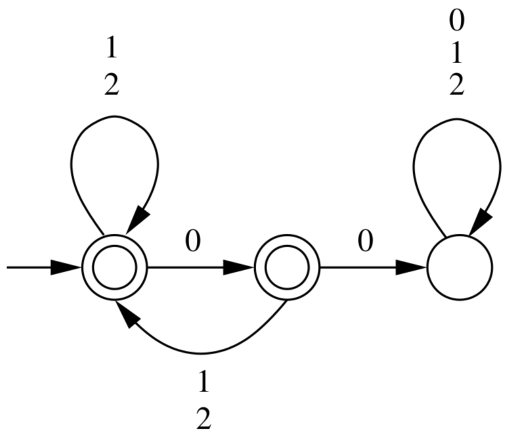
Solution:

The regular grammar $G = (\{S,T\},\ \{0,1,2\},\ S,\ R)$ for L_5 with the rules (R):

$$S \rightarrow \varepsilon \mid 0 \mid 0T \mid 1S \mid 2S$$
$$T \rightarrow 1S \mid 2S$$

(b) Design a deterministic finite automaton (DFA) to accept L_5. **(3)**

Solution:



Q6. Consider the language, $L_6 = \left\{a^{3k+1}b^{5k-2} \mid k \geq 1\right\} \subseteq \{a,b\}^*$.

(a) Write a context-free grammar (CFG) G with $\mathscr{L}(G) = L_6$. **(4)**

Solution:

The trick is to substitute $k = l+1$ and write L_6 as:

$$L_6 = \left\{a^{4+3l}b^{5l+3} \mid l \geq 0\right\}$$

Now it is easy to write a CFG $G = (\{S,T\},\ \{a,b\},\ S,\ R)$ for L_6 with the rules:

$$\begin{aligned} S &\rightarrow aaaaTbbb \\ T &\rightarrow \varepsilon \mid aaaTbbbbb \end{aligned}$$

Clearly, $\mathscr{L}(T) = \left\{a^{3l}b^{5l} \mid l \geq 0\right\}$. The rest is obvious.

(b) Design a push-down automaton (PDA) M with $\mathscr{L}(M) = L_6$.

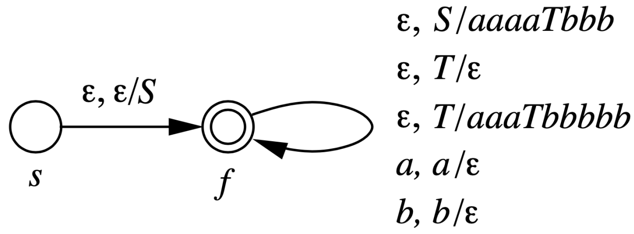
[

Hint: You may use CFG-to-PDA conversion procedure.] **(4)**

Solution:

A PDA can be designed for L_6 naïvely, that is, starting from the scratch.

Now that we have a CFG for L_6, it is easier to use the CFG-to-PDA conversion procedure to construct the following PDA with two states:



Q7. **(a)** Use a diagonalization argument to prove that the set of all infinite sequences of natural numbers is *uncountable*. **(3)**

Solution:

Let A be the set of all infinite sequences of natural numbers. Suppose that A is countable. Then there exists a bijective map $f : \mathbb{N} \to A$. Let us denote by $f(n)$ the sequence $a_{n_0}, a_{n_1}, a_{n_2}, \ldots$. We define an infinite sequence $b_0, b_1, b_2, \ldots, b_n, \ldots$ of natural numbers as follows:

$$b_n = \begin{cases} 2 & \text{if } a_{nn} = 1, \\ 1 & \text{if } a_{nn} \neq 1. \end{cases}$$

Since f is bijective, the sequence $b_0, b_1, b_2, \ldots$ is equal to $f(n)$ for some $n \in \mathbb{N}$. But $b_n \neq a_{n_n}$ by construction, that is, the sequence $b_0, b_1, b_2, \ldots$ is different from $f(n)$, a contradiction!

(b) Prove that the set of all finite sequences of natural numbers is *countable*. **(3)**

Solution:

Let C be the set of all finite sequences of natural numbers. We have $C = \bigcup_{n \in \mathbb{N}} C_n$, where C_n is the set of all sequences of natural numbers of length n. Since C_n can be viewed as the set $\mathbb{N}^n$ of all (ordered) n-tuples of natural numbers and since $\mathbb{N}^n$ is countable for every $n \in \mathbb{N}$, each C_n is countable. Therefore, C is the union of countably many countable sets and so is countable too.

— Question Paper Ends Here —