INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Stamp / Signature of the Invigilator

EXAMINATION (End Semester)													SEMESTER (Autumn 2024-2025)	
Roll Number										**Section**		**Name**		
Subject Number	C	S	6	0	0	0	5	**Subject Name**		*FOUNDATIONS OF COMPUTING SCIENCE*				
Department / Center of the Student										**Additional sheets**				

Important Instructions and Guidelines for Students

1. You must occupy your seat as per the Examination Schedule/Sitting Plan.
2. Do not keep mobile phones or any similar electronic gadgets with you even in the switched off mode.
3. Loose papers, class notes, books or any such materials must not be in your possession, even if they are irrelevant to the subject you are taking examination.
4. Data book, codes, graph papers, relevant standard tables/charts or any other materials are allowed only when instructed by the paper-setter.
5. Use of instrument box, pencil box and non-programmable calculator is allowed during the examination. However, exchange of these items or any other papers (including question papers) is not permitted.
6. Write on both sides of the answer script and do not tear off any page. **Use last page(s) of the answer script for rough work.** Report to the invigilator if the answer script has torn or distorted page(s).
7. It is your responsibility to ensure that you have signed the Attendance Sheet. Keep your Admit Card/Identity Card on the desk for checking by the invigilator.
8. You may leave the examination hall for wash room or for drinking water for a very short period. Record your absence from the Examination Hall in the register provided. Smoking and the consumption of any kind of beverages are strictly prohibited inside the Examination Hall.
9. Do not leave the Examination Hall without submitting your answer script to the invigilator. **In any case, you are not allowed to take away the answer script with you.** After the completion of the examination, do not leave the seat until the invigilators collect all the answer scripts.
10. During the examination, either inside or outside the Examination Hall, gathering information from any kind of sources or exchanging information with others or any such attempt will be treated as **'unfair means'**. Do not adopt unfair means and do not indulge in unseemly behavior.

Violation of any of the above instructions may lead to severe punishment.

Signature of the Student

To be filled in by the examiner

Question Number	1	2	3	4	5	6	7	8	9	10	Total
Marks Obtained											

Marks obtained (in words)	Signature of the Examiner	Signature of the Scrutineer

Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

Foundation of Computing Science (CS60005)		**Autumn Semester 2024-2025**
20-November-2024	**End-Semester Examination**	Maximum Marks: 100

Instructions:

- Write your answers in the question paper itself. Be brief and precise. Answer *all* questions.
- Write the answers only in the respective spaces provided. The last two blank pages may be used for rough work or leftover answers.
- In case you may need more space/pages, please ask for additional sheets in the exam hall and attach the same with this booklet while submitting.
- If you use any theorem / result / formula covered in the class, just mention it, do not elaborate. (unless the same thing has been explicitly asked to derive / prove in the question)
- Write all the proofs in mathematically / logically precise language. Unclear and/or dubious statements would be severely penalized.

— Question Paper Starts from Next Page —

Q1. Let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{Z}[i]$ the subset $\{a+ib \mid a,b \in \mathbb{Z}\}$ of $\mathbb{C}$. Elements of $\mathbb{Z}[i]$ are called *Gaussian integers*. For $z = x+iy \in \mathbb{C}$, we denote by $|z|$ the magnitude of z and by $\arg z$ the argument of z. Thus, $z = \sqrt{x^2+y^2}$ and $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$. We take $\arg z$ in the interval $[0, 2\pi)$.

Define a relation ρ on $\mathbb{C}$ as follows. Take $z_1, z_2 \in \mathbb{C}$. We say that $z_1 \; \rho \; z_2$ if and only if

either (i) $|z_1| < |z_2|$,

or (ii) $|z_1| = |z_2|$ and $\arg z_1 \leq \arg z_2$.

Moreover, define another relation σ on $\mathbb{C}$ as $z_1 \; \sigma \; z_2$ if and only if $|z_1| = |z_2|$. Answer the following.

(a) Prove that ρ is a partial order on $\mathbb{C}$. **(5)**

Solution:

(b) Prove that σ is an equivalence relation on $\mathbb{C}$. **(3)**

Solution:

(c) What are the equivalence classes of σ? (Provide a geometric description.) **(3)**

Solution:

Q2. Consider the context-free grammar G over $\{a,b\}$, with start symbol S, non-terminals $\{S,A,B\}$, and the following productions.

$$S \to aaB \mid Abb, \qquad A \to a \mid aA, \qquad B \to b \mid bB.$$

(a) What is the language, $\mathscr{L}(G)$, generated by G? **(2)**

Solution:

(b) Prove that this context-free grammar G is *ambiguous*. **(3)**

Solution:

(c) Design an *unambiguous* context-free grammar for $\mathscr{L}(G)$. **(3)**

Solution:

Q3. Let M be a Turing machine with one semi-infinite tape and two read/write heads. Each transition of M is determined by the current state p of the finite control, and the two symbols a and b scanned by the two heads. A transition of M is of the form $\delta(p,a,b) = (q,c,d,D_1,D_2)$ implying that the finite control goes to state q, the symbol a at the cell pointed by the first head is replaced by c, and the symbol b at the cell pointed by the second head is replaced by d. If both the heads point to the same tape cell ($a = b$ in this case), then the symbol at this cell is replaced by c (not by d unless $c = d$). Finally, the first head moves by one cell in direction D_1 (left or right), and the second head moves by one cell in direction D_2 (left or right).

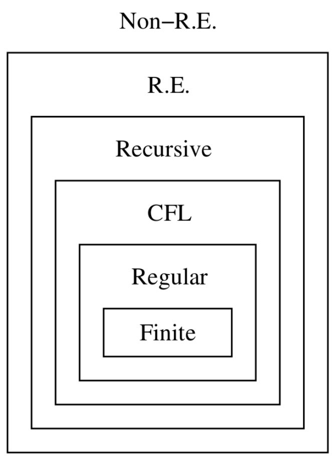
Argue that this two-head Turing machine M can be simulated by a standard Turing machine N with one semi-infinite tape and with only one read/write head. **(6)**

Solution:

Q4. Consider the hierarchy of languages shown in the adjacent figure (right). Place each of the following languages, L_1, L_2, L_3, L_4, L_5 (given in Parts (a)-(e), respectively) at the proper place in the hierarchy, that is, state and deduce whether the language L_i is:

- Finite
- Regular and Infinite
- Context-Free but Not Regular
- Recursive but Not Context-Free
- R.E. (recursively enumerable) but Not Recursive
- Non-R.E.

Provide clear and proper justifications/deductions for your judgments. No credits will be given for incorrect/incomplete/missing explanations, even though the truth value is correctly assigned.



In case of justifying Recursive or R.E. languages, construct appropriate Turing machines and/or supply appropriate reductions (note that, halting may be a choice of Turing machines, so do not use Rice's theorem).

(a) $L_1 = \mathscr{L}(G)$ where grammar G is defined over $\{a, b\}$, with start symbol S, non-terminals $\{S, T\}$, and the following productions. **(4)**

$$S \to \varepsilon \mid abTS, \qquad T \to \varepsilon \mid Tb.$$

L_1 is ____ .

Solution:

(b) The set L_2 of all strings $\alpha \in \{a, b, c\}^*$ containing an equal number of occurrences of a's, b's and c's. **(6)**

L_2 is ____ .

Solution:

(c) The complement of L_2 (from Part-(b)), that is, $L_3 = \{a,b,c\}^* \setminus L_2$. **(6)**

L_3 is ____ .

Solution:

(d) $L_4 = \{M \mid M$ is a Turing machine that halts on exactly 2024 input strings$\}$. **(6)**

L_4 is ____ .

Solution:

(e) $L_5 = \{M \mid M \text{ is a Turing machine that halts on at least 2024 input strings}\}$. **(6)**

L_5 is ____ .

Solution:

Q5. *Prove* or *disprove* the following statements (Parts (a)–(e)). Give clear justifications. No credits will be given for incorrect/incomplete/missing explanations, even though the truth value is correctly assigned.

(a) Prove/Disprove: Let C be an NP-complete problem. If $C \in$ co-NP, then NP = co-NP. **(6)**

Solution:

(b) Prove/Disprove: An $O(n^k)$ reduction algorithm from A to B followed by a deterministic $O(n^k)$ algorithm for B yields an $O(n^k)$ deterministic algorithm for A.
(Here n is the input size, and k is a positive integer constant > 1.) **(4)**

Solution:

(c) Prove/Disprove: The class P is closed under Kleene star, that is, if $L \in \mathrm{P}$, then $L^* \in \mathrm{P}$. **(8)**

Solution:

(d) Prove/Disprove: The class NP is closed under concatenation, that is, if $L_1, L_2 \in \text{NP}$, then $L_1 L_2 \in \text{NP}$. **(6)**

Solution:

(e) Prove/Disprove: The class NP-complete is closed under union, that is, if $L_1,\ L_2$ are NP-complete, then so is $L_1 \cup L_2$. **(6)**

Solution:

Q6. *Prove* or *disprove* the following assertions (Parts (a)-(b)). Give clear justifications. No credits will be given for incorrect/incomplete/missing explanations, even though the truth value is correctly assigned.

You may make use of the assumption that NP $\neq$ co-NP, if necessary. Clearly indicate where you require this assumption.

(a) Prove/Disprove: The following language is NP-complete.

$$\mathsf{SMALLCYCLE} = \left\{ \langle G \rangle \mid \text{The longest cycle in the directed graph } G \text{ is of length } \leq \left\lfloor \frac{n(G)}{2} \right\rfloor \right\}$$

(Here, $n(G)$ denotes the number of vertices in G and $\lfloor\ \rfloor$ is the floor function.) **(6)**

Solution:

(b) Prove/Disprove: The following language is NP-complete.

$$\mathsf{BIGCYCLE} = \left\{ \langle G \rangle \mid G \text{ is a directed graph having a cycle of length } \geq \left\lfloor \frac{n(G)}{2} \right\rfloor \right\}$$

(Here, $n(G)$ denotes the number of vertices in G and $\lfloor\ \rfloor$ is the floor function.) **(6)**

Solution:

Q7. An undirected graph is k-colorable if the vertices of the graph can be assigned a colour from a given fixed set of k colors such that no two adjacent vertices (sharing a common edge in between) receive the same color. Prove that, the language, $\mathsf{3COLOR} = \{\langle G \rangle \mid \text{ The undirected graph } G \text{ is 3-colorable}\}$, is PSPACE-complete if and only if PSPACE = NP.

(Note that, $\mathsf{3COLOR}$ is known to be NP-complete.) **(5)**

Solution:

— Question Paper Ends Here —