CS60005 : Foundations of Computing Science (Autumn 2024-2025)

Class Test 2

29-Oct-2024 (Tuesday) Maximum Marks: 20 06:00pm – 07:00pm

Roll: ____________________ **Name:** ____

[*Write your answers in question paper. Answer* *__all__* *questions. Be brief and mathematically / logically precise.*]

Q1. Let M be a Turing machine with $\mathscr{L}(M) = L$ and with exactly one accept state and exactly one reject state. Construct a Turing machine N by swapping the accept and reject states of M.

Prove or disprove: $\mathscr{L}(N) = \overline{L}$. **(5)**

Solution:

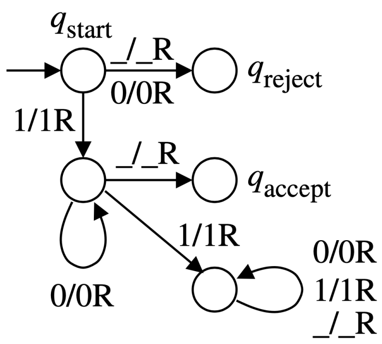
The assertion is *__false__* in general. Consider the pairwise disjoint languages:

$$L_a = \{\alpha \in \Sigma^* \mid M \text{ accepts } \alpha\}$$
$$L_r = \{\alpha \in \Sigma^* \mid M \text{ rejects } \alpha \text{ after halting}\}$$
$$L_l = \{\alpha \in \Sigma^* \mid M \text{ loops on } \alpha\}$$

Then $\mathscr{L}(M) = L_a$, $\mathscr{L}(N) = L_r$, and $L_a \cup L_r \cup L_l = \Sigma^*$. Unless $L_l = \phi$, we cannot say $\mathscr{L}(N) = \overline{L}$.

Q2. Prove or disprove: The language recognized by the Turing machine shown below is Turing-decidable.

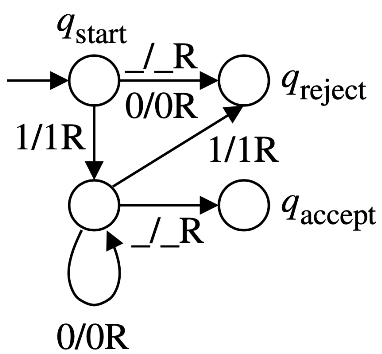
(5)



Solution:

The language, call it L, is Turing-decidable. In fact, L is the language of the regular expression 10^*. We know that regular languages are Turing-decidable.

[The given TM is not a decider for L, since it does not halt, for example, on the input 101. A decider for L is given below:



For solving this exercise, designing such a decider is not necessary.]

Q3. Prove that Turing-recognizable languages are closed under union. **(5)**

Solution:

Let $L_1 = \mathscr{L}(M_1)$ and $L_2 = \mathscr{L}(M_2)$ be two Turing-recognizable languages, where M_1 and M_2 are TMs. Let us design a TM M with $\mathscr{L}(M) = L_1 \cup L_2$. M runs M_1 and M_2 in "parallel". M may be designed as a two-tape machine. To start with M copies the input α to the second tape and positions both the heads at the beginning of the respective copies of the input. M simulates the behavior of M_1 on the first tape and that of L_2 on the second tape *simultaneously*. M accepts if and only if either the simulation of M_1 or that of M_2 accepts. If both M_1 and M_2 reject α after halting, then M also rejects. If one of M_1 and M_2 rejects after halting and the other by not halting, or if both M_1 and M_2 reject by not halting, then M rejects by not halting.

Q4. Let $L_1, L_2, \ldots, L_n$ be pairwise disjoint Turing-recognizable languages over the same alphabet Σ. Suppose that $\bigcup_{i=1}^{n} L_i = \Sigma^*$. Prove that each L_i is Turing-decidable. **(5)**

Solution:

From **Q3**, Turing-recognizable languages are closed under finite union. It follows that $\bigcup_{\substack{1 \le j \le n \\ j \neq i}} L_j$ is Turing-recognizable, that is, $\overline{L_i}$ is Turing-recognizable. Since L_i is also Turing-recognizable, it follows that L_i is Turing-decidable.

— End of Question Paper —