CS60005 : Foundations of Computing Science (Autumn 2024-2025)

Class Test 1

27-Aug-2024 (Tuesday) Maximum Marks: 20 06:00pm – 07:00pm

Roll: ____________________ **Name:** ____

[*Write your answers in question paper. Answer all questions. Be brief and mathematically / logically precise.*]

Q1. Let $z_1, z_2, \ldots, z_n$ be positive integers with $\sum_{i=1}^{n} z_i < 2^n - 1$. Prove that, there exist distinct disjoint non-empty subsets A, B of the set $\{z_1, z_2, \ldots, z_n\}$ with the property that $\sum_{a \in A} a = \sum_{b \in B} b$. **(4)**

Solution:

Let us define the set $S = \{z_1, z_2, \ldots, z_n\}$.

Since S has $2^n - 1$ non-empty subsets and the sum $\sum_{x \in Z} x$ for a non-empty subset Z of S can assume $\leq 2^n - 2$ values (between 1 and $2^n - 2$), by the pigeon-hole principle $\sum_{a \in A'} a = \sum_{b \in B'} b$ for two distinct non-empty subsets A' and B' of S.

Now, throw away the common elements from A' and B', that is, take $A = A' \setminus B' = A' \setminus (A' \cap B')$ and $B = B' \setminus A' = B' \setminus (A' \cap B')$. Then, A and B are non-empty too and we continue to have $\sum_{a \in A} a = \sum_{b \in B} b$.

Q2. From the semester examination results, the departmental head makes the following observations.

F_1 : *If Alice is meritorius, then Bob is not studious or Charles is not attentive.*

F_2 : *If Charles is attentive, then Diana is sincere.*

F_3 : *If Diana is sincere and Bob is studious, then Alice is meritorius.*

F_4 : *Bob is studious.*

You task is to answer the question: *"Is Charles attentive?"*

Frame (encode) the arguments logically, and formally deduce (applying logical inferencing rules) the answer being asked here. Present your solution as indicated in the following parts.

(a) Write all the propositions (that you will use to encode) with English statements (meaning). **(2)**

Solution:

We may use the following propositions.

a : Alice is meritorius.

b : Bob is studious.

c : Charles is attentive.

d : Diana is sincere.

(b) Write propositional logic formula to encode each of the *four* statements $F_1 - F_4$ given above. **(2)**

Solution:

The proposition logic encodings are as follows.

F_1 : $a \rightarrow (\neg b \vee \neg c)$

F_2 : $c \rightarrow d$

F_3 : $(d \wedge b) \rightarrow a$

F_4 : b

We *claim* that we shall derive the goal as, G : $\neg c$

– Question **Q2** continued to next page –

(c) Use logical inferencing rules to completely (and step-wise) derive the answer, and finally conclude whether Charles is attentive or not. **(4)**

Solution:

The logical deduction procedure is given in the following.

$$\begin{array}{ll} F_3: & (d \wedge b) \to a \\ F_4: & b \\ \hline \therefore G_1: & d \to a \end{array} \qquad \begin{array}{ll} F_1: & a \to (\neg b \vee \neg c) \\ F_4: & b \\ \hline \therefore G_2: & a \to \neg c \end{array}$$

$$\begin{array}{ll} G_1: & d \to a \\ G_2: & a \to \neg c \\ \hline \therefore G_3: & d \to \neg c \end{array} \qquad \begin{array}{ll} F_2: & c \to d \\ G_3: & d \to \neg c \\ \hline \therefore G: & \neg c \end{array}$$

Conclusion: *Charles is not attentive.*

(NO marks is given only for the conclusion without logical derivation/inferencing.)

Q3. Let $\mathbb{A}$ be one of the sets $\mathbb{N}$ (the set of natural numbers including zero), $\mathbb{Z}$ (the set of integers), $\mathbb{Q}$ (the set of rationals), or $\mathbb{R}$ (the set of reals). A function $\varphi : \mathbb{A} \to \mathbb{A}$ is called *monotonic increasing* if $m \leq n$ implies $\varphi(m) \leq \varphi(n)$. It is called *strictly monotonic increasing* if $m < n$ implies $\varphi(m) < \varphi(n)$.

(a) Show that a strictly monotonic increasing function is injective. **(2)**

Solution:

Let $f(m) = f(n)$.

If $m < n$, we have $f(m) < f(n)$, whereas if $m > n$, we have $f(m) > f(n)$.

So, we must have $m = n$.

– Question Q3 continued to next page –

(b) Give an example of a function $\mathbb{N} \to \mathbb{N}$ that is injective but not strictly monotonic increasing. **(2)**

Solution:

Consider $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ n & \text{if } n \geq 2. \end{cases}$

(c) Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function with the property that $|f(n+1) - f(n)| \leq 1$ for all $n \in \mathbb{N}$. Show that f is strictly monotonic increasing. **(4)**

Solution:

If $f(n+1) - f(n) = 0$ for some n, then f is not injective. So assume that, $f(n+1) - f(n) \in \{1, -1\}$ for all n. We can now claim that $f(n+1) - f(n) = 1$ for all n, which eventually proves that f is strictly monotonic increasing.

For proving the claim, assume that there exists an n, say N, for which $f(N+1) - f(N) = -1$. Let $k = f(N)$, so that $f(N+1) = k - 1$. Now, $f(N+2) = f(N+1) \pm 1$. Since f is given to be injective, this forces $f(N+2) = (k-1) - 1 = k - 2$. Similarly, we have $f(N+3) = k - 3$, $f(N+4) = k - 4$ and so on. In particular, $f(N+k-1) = 1$ and $f(N+k) = 0$. Since $f(N+k+1)$ must be non-negative and differ from $f(N+k)$ by (at most) 1, we must then have $f(N+k+1) = 1 = f(N+k-1)$, a contradiction to the injectivity of f.

— End of Question Paper —