Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

Foundations of Computing Science (CS60005) **Autumn Semester, 2022-2023**

Mid-Semester Examination **22-Sep-2022 (Thursday), 09:00–11:00** **Maximum Marks: 60**

Instructions:

- *Write your answers in the answer booklet provided to you in the examination hall.*
- *There are a total of SIX questions, each having* 10 *marks.*
- *Answer ALL the questions (or as many as you can) mentioning the question numbers clearly.*
- *Be brief and precise. Write the answers for all parts of a question together.*
- *If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.*
- *Write all the proofs/deductions in mathematically/logically precise language. Unclear and/or dubious statements would be severely penalized.*

Q1. Your task is to (logically) solve a murder-mystery on behalf of Sherlock Holmes which appeared in the novel "A Study in Scarlet" by Sir Arthur Conan Doyle. The arguments (simplified from the novel) go as follows.

F_1 : There was a murder. If it was not done for robbery, then either it was a political assassination, or it might be for a woman.

F_2 : In case of robbery, usually something is taken.

F_3 : However, nothing was taken from the murderer's place.

F_4 : Political assassins leave the place immediately after their assassination work gets completed.

F_5 : On the contrary, the assassin left his/her tracks all over the murderer's place.

F_6 : For an assassin, to leave tracks all over the murderer's place indicates that (s)he was there all the time (for long duration).

Your goal is to (logically) find the reason for the murder. Please frame the above arguments logically (using propositional logic) and formally derive the solution. *Present your answer as asked in the following parts.*

(a) Write all propositions with English meaning (statements) that you have used. **(2)**

(b) Build suitable propositional logic formula to encode each of the *six* statements above. **(3)**

(c) Show all deduction steps (with the name of the rules you apply) to derive the goal (mystery). **(4)**

(d) Conclude what was the reason for the murder. **(1)**

Q2. Let the relation σ on $\mathbb{N}$ (the set of natural numbers) consist only of the following tuples:

$$\sigma = \Big\{(n,n) \mid n \in \mathbb{N}\Big\} \bigcup \Big\{(2n, 2n-1) \mid n \in \mathbb{N}\Big\} \bigcup \Big\{(2n, 2n+1) \mid n \in \mathbb{N}\Big\}.$$

(a) Prove that σ is a partial order. **(6)**

(b) Is σ a total (that is, linear) order? **(2)**

(c) Is $\mathbb{N}$ a lattice under σ? **(2)**

Q3. Let S be the set of all infinite bit sequences. In the class, S has been proven to be uncountable (using diagonization argument). The n-th element of a sequence $\alpha \in S$ is denoted by $\alpha(n)$ for $n \geq 0$.

Prove the countability / uncountability of each of the following subsets of S.

(a) $T_1 = \Big\{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0, \text{ for } \underline{\text{some}}\ n \geq 0\Big\}$. **(5)**

(b) $T_2 = \Big\{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0, \text{ for } \underline{\text{no}}\ n \geq 0\Big\}$. **(5)**

Note: Solve the above parts independently, that is, do not use the result of any part in the other.

Q4. Let $R = \mathbb{Z} \times \mathbb{Z}$ (Cartesian product between set of integers). Define addition and multiplication on R as:

$$\begin{aligned} a,b) + (c,d) &= (a+c,\ b+d), \text{ and} \\ (a,b) \cdot (c,d) &= (ac + ad + bc,\ 2ac + bd). \end{aligned}$$

(a) Verify that R is a ring under these two operations. **(7)**

(b) Prove that R is a commutative ring with unity. **(3)**

Q5. Let L be a language over an alphabet Σ. Recall that a string x is called a prefix of a string y if $y = xz$ for some string z. For example, all the prefixes of $abbab$ are $\varepsilon, a, ab, abb, abba, abbab$. From L, we generate the language $\texttt{dupPrefix}(L)$ by duplicating prefixes of strings in L. More precisely, we define,

$$\texttt{dupPrefix}(L) = \Big\{xy \mid y \in L, \text{ and } x \text{ is a prefix of } y\Big\}.$$

Prove / Disprove:

(a) If L is regular, then $\texttt{dupPrefix}(L)$ must also be regular. **(5)**

(b) If L is not regular, then $\texttt{dupPrefix}(L)$ must also be non-regular. **(5)**

Q6. Consider the following language L_1 over the alphabet $\{a, b, \#\}$.

$$L_1 = \Big\{x\#y \mid x, y \in \{a,b\}^*,\ x \neq y,\ |x| = |y|\Big\}.$$

Here, $|w|$ denotes the length of the string w. Prove / Disprove: L_1 is context-free. **(10)**