Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

Foundations of Computing Science (CS60005) **Autumn Semester, 2022-2023**

End-Semester Examination **Date: 18-Nov-2022, 2:00 PM – 5:00 PM** **Marks: 100**

Instructions:

- Write your answers in the answer booklet provided to you in the examination hall.
- There are a total of FIVE questions, each having 20 marks.
- Answer ALL the questions (or as many as you can) mentioning the question numbers clearly.
- Be brief and precise. Write the answers for all parts of a question together.
- State any results you use or assumptions you make.
- Write all the proofs/deductions in mathematically/logically precise language.
- Sketchy proofs or claims without reasoning receive no credit.

1. Only write down all the correct choice(s) / option(s) for each of the following questions. $\boxed{10 \times 2 = 20}$

   Note: *No justification is required for your given option(s); but a penalty of $\frac{1}{2}$ per question will be deducted for guessing wrong, including not making all choices (in case of multiple correct options).*

   (a) The propositional logic statement $\big[(p \vee q \vee r) \rightarrow s\big]$ is equivalent to which one/more of the following?

   (A) $\neg s \rightarrow (p \vee q \vee r)$
   (B) $\neg s \rightarrow (p \wedge q \wedge r)$
   (C) $(p \rightarrow s) \wedge (q \rightarrow s) \wedge (r \rightarrow s)$
   (D) $(p \rightarrow s) \vee (q \rightarrow s) \vee (r \rightarrow s)$

   (b) According to political experts, "A person who is a radical (*radical*) is elected (*elected*) if (s)he is conservative (*conservative*), but otherwise is not elected". Which of the following is/are the correct logical representation(s) of the above statement made by political experts?

   (A) $\big(radical \wedge elected\big) \leftrightarrow conservative$
   (B) $radical \rightarrow \big(elected \leftrightarrow conservative\big)$
   (C) $radical \rightarrow \big((conservative \rightarrow elected) \vee \neg elected\big)$
   (D) $\big(conservative \rightarrow (radical \wedge elected)\big) \vee \neg elected$

   (c) Let P and Q are two predicates, and $F_1 \Leftrightarrow F_2$ symbolically denotes equivalence of two predicate logic formulas, F_1 and F_2 (that is, F_1 if and only if F_2). Which of the following is/are incorrect?

   (A) $\forall x\ \big[P(x) \vee Q(x)\big] \Leftrightarrow \forall x\ P(x) \vee \forall x\ Q(x)$
   (B) $\exists x\ \big[P(x) \vee Q(x)\big] \Leftrightarrow \exists x\ P(x) \vee \exists x\ Q(x)$
   (C) $\forall x\ \big[P(x) \wedge Q(x)\big] \Leftrightarrow \forall x\ P(x) \wedge \forall x\ Q(x)$
   (D) $\exists x\ \big[P(x) \wedge Q(x)\big] \Leftrightarrow \exists x\ P(x) \wedge \exists x\ Q(x)$

   (d) Let X, Y, Z be sets. Which of the following formulas is/are wrong?

   (A) $(X \cap Y \cap Z) \setminus Z = \emptyset$

(B) $(X \cup Y \cup Z) \setminus Z = X \cup Y$
(C) $(X \cap Y) \setminus Z = (X \setminus Z) \cap (Y \setminus Z)$
(D) $(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z)$

(e) Let $A = \{a, b, c, d\}$. How many binary operations f on A are there such that $f(a, b)$ is either c or d?
(A) 15^4
(B) 4^{15}
(C) 2×4^{15}
(D) 2×15^4

(f) Define a relation ρ on $\mathbb{Z}$ (the set of integers) such that $a \ \rho \ b$ if and only if either $a = b$ or $|a - b| > 1$. Which of the following statements about ρ is/are incorrect?
(A) ρ is reflexive
(B) ρ is symmetric
(C) ρ is antisymmetric
(D) ρ is transitive

(g) Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denotes the set of natural numbers, integers and rational numbers, respectively. Which of the following sets is/are <u>countable</u>?
(A) $\mathbb{N} \times \mathbb{N}$
(B) $\mathbb{Z} \times \mathbb{Q}$
(C) $\mathbb{Q} \setminus \mathbb{N}$
(D) $\mathbb{Q} \times \mathbb{Q}$

(h) What is the inverse of an element a in the group $G = \{a \in \mathbb{R} \mid a > 0\}$ ($\mathbb{R}$ denotes the set of real numbers) under the operation $\odot$ defined by $a \odot b = a^{\ln b}$?
(A) $1/a$
(B) $1/e^{\ln a}$
(C) $e^{1/\ln a}$
(D) $1/\ln a$

(i) Consider the set $\mathbb{Z}$ of all integers with the following operations, where s and t are constant integers.

$$\begin{aligned} a \oplus b &= a + b + s, \\ a \odot b &= a + b + tab. \end{aligned}$$

Which of the following is/are a necessary and sufficient condition for $(\mathbb{Z}, \oplus, \odot)$ to be a ring?
(A) $st = 1$.
(B) $s = t = -1$.
(C) s and t can have any values.
(D) s and t can have any values with $t \neq 0$.

(j) Which of the following functions $f : \mathbb{Z} \to \mathbb{Z}$ ($\mathbb{Z}$ denotes the set of integers) is/are a homomorphism of the ring $(\mathbb{Z}, +, \cdot)$ to itself?
(A) $f(x) = 1$
(B) $f(x) = x$
(C) $f(x) = 2x$
(D) $f(x) = x^2$

2. For each of the following languages, identify whether the language is recursive, *r.e.* but not recursive or not *r.e.* For all languages defned below, $\mathcal{M}$ denotes a Turing machine. Justify your answer.

   (a) $\{\mathcal{M} \mid \mathcal{M}$ on input x moves its head more than 100 cells away from the left endmarker$\}$.

   (b) $\{\mathcal{M} \mid L(\mathcal{M})$ is recursive$\}$.

   (c) $\{\mathcal{M} \mid L(\mathcal{M})$ has atleast 100 states$\}$.

   (d) $\{\mathcal{M} \mid \mathcal{M}$ halts on all inputs of length less than 100$\}$.

$4 \times 5 = 20$

3. (a) Let ϕ be a Boolean formula in CNF. An assignment to the variables of ϕ is called *not-all-equal* if in each clause, at least one literal is assigned 1 and at least one literal is 0. Show that the following language is **NP**-Complete.

$$\mathsf{NESAT} = \{\phi \mid \phi \text{ is a CNF formula having a satisfying } \textit{not-all-equal} \text{ assignment}\}$$

   Assume no constants are allowed in the formulae. 10

   (b) Let S be a finite set and $\mathcal{C} = \{C_1, C_2, ..., C_k\}$ be a collection of subsets of S, for some $k > 0$. We say S is *2-colourable with respect to* $\mathcal{C}$ if we can colour each element of S in red or blue, such that every C_i contains at least one red element and at least one blue element. Show that the language

$$\mathsf{2COLOUR} = \{\langle S, \mathcal{C}\rangle \mid S \text{ is 2-colourable with respect to } \mathcal{C}\}$$

   is **NP**-Complete. 10

4. Answer true or false. Justify.

   (a) If L_1, L_2 are **NP**-complete, then so is $L_1 \cup L_2$.

   (b) All **NP**-Hard problems are decidable.

   (c) $\mathbf{NSPACE}(2^n) \subseteq \mathbf{DSPACE}(4^n)$.

   (d) **PSPACE** is closed under Kleene star (i.e., if $L \in \mathbf{PSPACE}$, then $L^* \in \mathbf{PSPACE}$).

   (e) $\mathbf{polyL} \neq \mathbf{polyNL}$, where $\mathbf{polyL} = \cup_{c>0}\mathbf{DSPACE}(\log^c n)$, $\mathbf{polyNL} = \cup_{c>0}\mathbf{NSPACE}(\log^c n)$.

$5 \times 4 = 20$

5. A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two sets V_1, V_2 such that every edge in E has one end-point in V_1 and the other in V_2. Let $\mathsf{BIPARTITE}$ be the set of all undirected bipartite graphs. Show that $\mathsf{BIPARTITE} \in \mathbf{NL}$.
   Hint: Use the fact that $\mathbf{coNL} = \mathbf{NL}$. 20