Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

Foundations of Computing Science (CS60005)		**Autumn Semester, 2022-2023**
Class Test 1	**01-Sep-2022 (Thursday), 17:30–18:30**	**Maximum Marks: 20**

Instructions:

- Write your answers in the answer booklet provided to you in the examination hall.
- There are a total of THREE questions, having 6 marks, 7 marks and 7 marks, respectively.
- Answer ALL the questions (or as many as you can) mentioning the question numbers clearly.
- Be brief and precise. Write the answers for all parts of a question together.
- If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.
- Write all the proofs/deductions in mathematically/logically precise language.
  Unclear and/or dubious statements would be severely penalized.

– The question paper starts from the next page –

Q1. Let $\texttt{Cat}(x)$, $\texttt{Dog}(x)$, $\texttt{Striped}(x)$, $\texttt{Friends}(x,y)$, $\texttt{Equal}(x,y)$ and $\texttt{ShortTempered}(x)$ be predicates to be evaluated over the set of all animals. The predicates have the following obvious interpretations.

- $\texttt{Cat}(x)$ $\big($or $\texttt{Dog}(x)\big)$ evaluates to $\texttt{True}$ iff x is a cat $\big($or dog$\big)$.
- $\texttt{Striped}(x)$ evaluates to $\texttt{True}$ iff the coat of x is striped.
- $\texttt{Friends}(x,y)$ evaluates to $\texttt{True}$ iff x and y are friends.
  Obviously, $\texttt{Friends}(x,y)$ also implies $\texttt{Friends}(y,x)$.
- $\texttt{Equal}(x,y)$ evaluates to $\texttt{True}$ iff x and y are one and the same animal.
- $\texttt{ShortTempered}(x)$ evaluates to $\texttt{True}$ iff x is short-tempered.

Express the following English language sentences as predicate logic sentences (formulae without free variables). You may assume that the domain for evaluating the truth of these sentences is always the set of all animals (which could include animals other than cats and dogs as well).

(a) There is a short-tempered dog who is not friendly with any other dog, but is friendly with at least one striped cat. **(2)**

(b) Every cat that is not striped is friendly with at least one dog that is not a friend of any striped cat. **(2)**

(c) Every short-tempered cat is friendly with one and only one striped dog. **(2)**

Solution:

(a)
$$\exists x \Big[\Big(\texttt{Dog}(x) \wedge \texttt{ShortTempered}(x)\Big) \bigwedge \forall y \Big(\texttt{Dog}(y) \wedge \neg\texttt{Equal}(x,y) \to \neg\texttt{Friends}(x,y)\Big) \bigwedge \exists z \Big(\texttt{Cat}(z) \wedge \texttt{Striped}(z) \wedge \texttt{Friends}(x,z)\Big)\Big]$$

(b)
$$\forall x \Big[\Big(\texttt{Cat}(x) \wedge \neg\texttt{Striped}(x)\Big) \to \exists y \Big\{\Big(\texttt{Dog}(y) \wedge \texttt{Friends}(x,y)\Big) \bigwedge \forall z \Big(\texttt{Cat}(z) \wedge \texttt{Striped}(z) \to \neg\texttt{Friends}(y,z)\Big)\Big\}\Big]$$

(c)
$$\forall x \Big[\Big(\texttt{Cat}(x) \wedge \texttt{ShortTempered}(x)\Big) \to \exists y \Big\{\Big(\texttt{Dog}(y) \wedge \texttt{Striped}(y) \wedge \texttt{Friends}(x,y)\Big) \bigwedge \forall z \Big(\texttt{Dog}(z) \wedge \texttt{Striped}(z) \wedge \neg\texttt{Equal}(y,z) \to \neg\texttt{Friends}(x,z)\Big)\Big\}\Big]$$

Q2. Let $\mathbb{Z}$ be the set of all integers. Define a relation R on $\mathbb{N}$ (the set of positive integers) as follows:

$$\forall a, b \in \mathbb{N},\ a\,R\,b \text{ if and only if } \exists i \in \mathbb{Z},\ \frac{a}{b} = 2^i.$$

(a) Prove that R is an equivalence relation. **(3)**

(b) List the equivalence classes defined by R on $\mathbb{N}$. **(2)**

(c) Prove / Disprove: R is a partial order. **(2)**

Solution:

(a) We verify the three propoerties of equivalence relation as follows.

[Reflexive] For $a \in \mathbb{N}$, $\frac{a}{a} = 1 = 2^0$, so $a\,R\,a$.

[Symmetric] Suppose that $a\,R\,b$. Then, we have

$$\begin{aligned} a\,R\,b &\Rightarrow \exists i \in \mathbb{Z},\ \frac{a}{b} = 2^i \\ &\Rightarrow \exists i \in \mathbb{Z},\ \frac{b}{a} = 2^{-i} \\ &\Rightarrow b\,R\,a, \quad \text{because} \quad -i \in \mathbb{Z}. \end{aligned}$$

[Transitive] Suppose that $a\,R\,b$ and $b\,R\,c$. By definition,

$$\begin{aligned} a\,R\,b &\Rightarrow \exists i \in \mathbb{Z},\ \frac{a}{b} = 2^i, \\ b\,R\,c &\Rightarrow \exists j \in \mathbb{Z},\ \frac{b}{c} = 2^j. \end{aligned}$$

But then $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = 2^{i+j}$. Since $i + j \in \mathbb{Z}$, we have $a\,R\,c$.

(b) Given any natural number x, there exist $k \in \mathbb{N}$ and $i \in \mathbb{N} \cup \{0\}$ such that k is not divisible by 2, and $x = k \times 2^i$. For any natural number $y = k \times 2^j$, we have $\frac{x}{y} = 2^{i-j}$, where $(i - j)$ is an integer. Thus, $x\,R\,y$, implying that x and y belong to the same equivalence class.

Conversely, if $x = k \times 2^i$ and $y = l \times 2^j$ for odd k, l with $k \neq l$, then $\frac{x}{y}$ is not of the form 2^t with $t \in \mathbb{Z}$. Therefore the equivalence classes of R are as follows:

$$\begin{gathered} \{1 \times 2^i \mid i = 0, 1, 2, \ldots\} \\ \{3 \times 2^i \mid i = 0, 1, 2, \ldots\} \\ \{5 \times 2^i \mid i = 0, 1, 2, \ldots\} \\ \{7 \times 2^i \mid i = 0, 1, 2, \ldots\} \\ \vdots \end{gathered}$$

(c) *False.*

R is not a partial order as *anti-symmetry* does not hold.

As a counter-example, $\frac{4}{2} = 2^1$ and $\frac{2}{4} = 2^{-1}$. However, $2 \neq 4$.

Q3. Consider the real intervals $(0, 1)$ and $[0, 1]$. A function $f : (0, 1) \to [0, 1]$ is defined as follows. Take $x \in (0, 1)$. Find (the unique) $n \in \mathbb{N}$ such that $\frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}$. Define $f(x) = \frac{3-2^n x}{2^n}$.

(a) Prove / Disprove: f is injective. **(4)**

(b) Prove / Disprove: f is surjective. **(3)**

Solution:

(a) *True*.

Break the domain of f into mutually disjoint intervals $I_n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$ for all $n \in \mathbb{N}$.
We have $(0, 1) = \bigcup_{n \in \mathbb{N}} I_n$.

The codomain, on the other hand, can be decomposed as $[0, 1] = \{0\} \bigcup \left(\bigcup_{n \in \mathbb{N}} J_n \right)$, where $\{0\}$ and the intervals $J_n = \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ for $n \in \mathbb{N}$ are again mutually disjoint.

For each $n \in \mathbb{N}$, the function f maps I_n to J_n, because –

$$
\begin{aligned}
x \in I_n \quad &\iff \quad \frac{1}{2^n} \leq x < \frac{1}{2^{n-1}} \\
&\iff \quad -\frac{1}{2^{n-1}} < -x \leq -\frac{1}{2^n} \\
&\iff \quad \frac{3}{2^n} - \frac{1}{2^{n-1}} < \frac{3}{2^n} - x \leq \frac{3}{2^n} - \frac{1}{2^n} \\
&\iff \quad \frac{1}{2^n} < \frac{3 - 2^n x}{2^n} \leq \frac{1}{2^{n-1}} \\
&\iff \quad f(x) \in J_n.
\end{aligned}
$$

Moreover, the restriction of f to $I_n \to J_n$ is bijective, because –

(i) $f(x) = f(y)$ for $x, y \in I_n$ implies $x = y$, and
(ii) for each $y \in J_n$, we have $x = \frac{3-2^n y}{2^n} \in I_n$ such that $f(x) = y$.

It follows that f is injective.

(b) *False*.

We have,

$$f((0,1)) = f\left(\bigcup_{n \in \mathbb{N}} I_n \right) = \bigcup_{n \in \mathbb{N}} J_n = [0, 1] \setminus \{0\},$$

that is, 0 is not in the range of f.
It follows that f is *not* surjective.