Let A and B be uncountable sets with $A \subseteq B$.
Prove or disprove: A and B are equinumerous. $\rightarrow$ FALSE

Solution:

$A = \mathbb{R} = c \qquad B = 2^{\mathbb{R}} = 2^{c} > c$ (1)

uncountable

$|A| < |B|$

false

Let A be an uncountable set and B a countably infinite subset of A.
Prove or disprove: A is equinumerous with A – B. → TRUE

$\because \; |A-B| \leq |A|$

To prove, $|A| \leq |A-B|$?

$\exists \, f : A \to (A-B)$ injective

$B = \{b_1, b_2, b_3 \ldots\ldots\}$

$C = \{c_1, c_2, c_3, \ldots\ldots\}$

$c_i \to c_{2i}$

$b_i \to c_{2i-1}$

C countably infinite

$C \subseteq A - B$

? $\boxed{a \in B}$

$$f \begin{cases} a \mapsto a & \boxed{\begin{array}{l} a \notin C \\ a \notin B \end{array}} \\ a \mapsto c_{2n} & \boxed{a \in C} \\ a \mapsto c_{2n-1} & a \in B \end{cases}$$

injective

Prove that the real interval [0, 1) is equinumerous with the unit square [0, 1) × [0, 1).

[where, $R = [0,1)$]

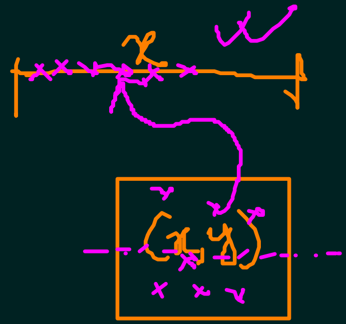
(i) $f : R \to R \times R$ is injective $\Rightarrow |R| \leq |R \times R|$

$|R| = |R \times R|$?

(ii) $g : R \times R \to R$ is injective $\Rightarrow |R \times R| \leq |R|$

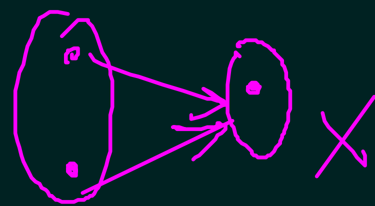
(i) $0.r_1 r_2 r_3 \ldots\ldots \longmapsto (0.r_1 r_2 r_3 \ldots\ldots, 0)$

injective map.



(ii) $(0.a_1 a_2 a_3 \ldots, 0.b_1 b_2 b_3 \ldots) \longmapsto \underbrace{0.a_1 b_1 a_2 b_2 a_3 b_3}_{R} \ldots$

injective map



Corollary

$|\mathbb{R}| = |\mathbb{C}|$

$[0,1)$ $\quad$ $(x+iy)$

Let Z[x] denote the set of all univariate polynimials with integer coefficients. Prove that Z[x] is countable.

x

$$Z[x] = \{ d_0 + d_1 x + d_2 x^2 + \cdots \mid d_i \in \mathbb{Z} \}$$

$\aleph_0$

$\aleph_0^{n} \rightarrow n \geq 0 \rightarrow \in \mathbb{N}$

$\aleph_0 \times \aleph_0 \times \aleph_0 \cdots$ Countable number of cross products

✓

Define an operation $\circ$ on $G = \mathbb{R}^* \times \mathbb{R}$ as $(a,b)\circ(c,d)=(ac, bc+d)$.
Prove that, $(G,\circ)$ is a non-abelian group.

$\mathbb{R}^* = \mathbb{R}\setminus\{0\}$

- Closure : $a \in \mathbb{R}^*, c \in \mathbb{R}^* \Rightarrow ac \in \mathbb{R}^*$

  $bc \in \mathbb{R} \quad d \in \mathbb{R} \Rightarrow bc+d \in \mathbb{R}$

- Associative : $(a,b)\circ[(c,d)\circ(e,f)] = [(a,b)\circ(c,d)]\circ(e,f)$ ✓

- Identity : $(a,b)\circ\boxed{(1,0)} = (a,b)$

  $\boxed{(1,0)}\circ(a,b) = (a,b)$

- Inverse : $(a,b)^{-1} = ?$

  $\left(\frac{1}{a}, -\frac{b}{a}\right)$

$(a,b)\circ(x,y) = (a,b)$

$\begin{cases} ax = a \\ bx+y = b \end{cases}$

$(a,b)\circ\cancel{(a,b)^{-1}} = (1,0)$

(x,y)

$\boxed{ax = 1 \oplus bx+y = 0}$

* abelian : ✓ NO

(Commutative)

$(3,2)\circ(3,5) \neq (3,5)\circ(3,2)$

Let G be a (multiplicative) group, and H, K are subgroups of G. Prove that,

(a) H∩K is a subgroup of G.

(b) H∪K is a subgroup of G if and only if H⊆K or K⊆H.

(c) Define HK = {hk | h∈H, k∈K}. Define KH analogously.
Prove that, HK is a subgroup of G if and only if HK = KH.

(a) $\left\{ \begin{array}{l} (H \cap K) \text{ is} \\ \text{also group} \end{array} \right\}$ To Prove

① c1 : $a, b \in H \cap K \Rightarrow a \in H \wedge a \in K$, $b \in H \wedge b \in K$

$\Rightarrow ab \in H \wedge ab \in K \Rightarrow ab \in H \cap K$

② — assoc : $(ab)(c) = (a)(bc)$ ✓

③ — identity : $e \in H \wedge e \in K \Rightarrow e \in H \cap K$

④ — inverse : $a \in H \cap K \Rightarrow a \in H \wedge a \in K \Rightarrow a^{-1} \in H \wedge a^{-1} \in K$

$\Rightarrow a^{-1} \in H \cap K$

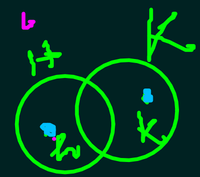
(b) "⇐" $H \subseteq K \Rightarrow H \cup K = K$

"⇒" $(H \cup K)$ subgroup but $\underbrace{H \not\subseteq K \text{ and } K \not\subseteq H}$ ✓ ?

$\left. \begin{array}{l} \exists h \in H \\ \exists k \in K \end{array} \right\}$ but $h \notin K$, but $k \notin H$

$a \to b$

$(a \wedge \neg b)$ ✓



$hk \in H \cup K$

$\Rightarrow hk \in H \vee hk \in K$

If $hk \in H$ $h \in H \Rightarrow h^{-1} \in H$

$\underbrace{h^{-1}}_{\in H} \underbrace{(hk)}_{\in H} = k \in H$ ✓

$hk \in K$ $k \in K$, $k^{-1} \in K$

$\Rightarrow h \in K$

$HK = \{hk \mid h \in H, k \in K\}$ ▶ HK is subgroup iff $HK = KH$

$KH = \{kh \mid h \in H, k \in K\}$

(C) "$\Rightarrow$" HK is a subgroup of G $\quad hk \in HK \rightsquigarrow hk \in KH$

$y =$

H, K are

$(hk)^{-1} \in HK \Rightarrow k^{-1}h^{-1} \in HK$ $\quad HK \subseteq KH$

$y = k^{-1}h^{-1} \in KH$

— do $\longrightarrow$? $KH \subseteq HK$?

A B

"$\Leftarrow$" $HK = KH$ (given) $\quad$ ~~$a_i \leftarrow hk$ closed $(G, *)$~~ $\quad a_i$ b_j

associative:

$h_1 k_1 = k_2 h_2$

$[(h_1k_1)[(h_2k_2)](h_3k_3)] = (h_1 h_2' k_1' k_2)(h_3 k_3)$

① $h_2' k_1'$ $\qquad h_3' k_2'$

$h_2' k_1'$

$(h_1 k_1)(h_2 k_2)$ $\qquad = (h_1 h_2' h_3'' k_3'' k_2' k_3)$

② clos: $= \underbrace{h_1 h_2'}_{H} \underbrace{k_1' k_2}_{K}$

id $e \in H \quad e \in K \Rightarrow e \in HK$

inv: $x = hk \in HK \quad x^{-1} = (hk)^{-1} = k^{-1}h^{-1} \in KH = HK$

Let R = $\mathbb{Z}\times\mathbb{Z}$, and r, s be constant integers. Define two operations, on R as follows:
$(a,b)+(c,d) = (a+c,b+d)$ and $(a,b)*(c,d) = (ad+bc+rac,bd+sac)$.
Prove that, R is a ring under these operations, + and $*$.

Solution ① $(a,b)+(c,d) = (c,d)+(a,b)$ ✓

7 properties

②

Let $R_1, R_2, \ldots, R_n$ be rings. Prove that, the Cartesian product $(R_1 \times R_2 \times \cdots \times R_n)$ is a ring under component-wise addition and multiplication.
Show that, if each R_k is a ring with identity, then so also is the product.

— do yourself —