Let $A, B, C \in \mathcal{U}$ are three arbitrary sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Prove that, $B = C$.

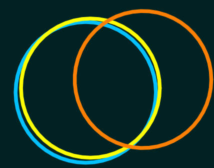
$$
\begin{aligned}
B &= B \cup (A \cap B) \\
&= B \cup (A \cap C) = (A \cup B) \cap (B \cup C) \\
&= (A \cup C) \cap (B \cup C) \\
&= (A \cap B) \cup C \\
&= (A \cap C) \cup C = C
\end{aligned}
$$

any $x \in B$

$B \subseteq C$ + $C \subseteq B$

that $x \in C$

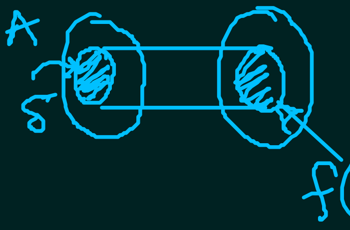
very



$$
\begin{aligned}
B &= B \cap (A \cup B) = B \cap (A \cup C) \\
&= (A \cap B) \cup (B \cap C) \\
&= (A \cap C) \cup (B \cap C) \\
&= (A \cup B) \cap C \\
&= (A \cup C) \cap C = C
\end{aligned}
$$

For a function $f : A \to B$, define a function $\mathscr{F} : \mathscr{P}(A) \to \mathscr{P}(B)$ as $\mathscr{F}(S) = f(S)$ for all $S \subseteq A$. Prove that:
(a) $\mathscr{F}$ is injective if and only if f is injective. **(b)** $\mathscr{F}$ is surjective if and only if f is surjective.

$f(S) = \{ f(s) \mid s \in S \}$



$f(a_1) = b_1$

$f(a_2) = b_2$

(a) "$\Rightarrow$" $\mathcal{F}$ is injective (one-to one)

means, $\mathcal{F}(S_1) = \mathcal{F}(S_2) \Rightarrow S_1 = S_2$ (given)

$f(S_1) = f(S_2) \xrightarrow[\text{P.T}]{} S_1 = S_2$ } f is injective.

$\Rightarrow \mathcal{F}(S_1) = \mathcal{F}(S_2) \Rightarrow$

"$\Leftarrow$" DO IT YOURSELF!

(b) "$\Leftarrow$" f is surjective. [given] $\forall b \in B$, $\exists a \in A$, s.t. $f(a) = b$.

$\Updownarrow$

$S_2 \subseteq B$ $\exists S_1 \in A$ s.t. $f(S_1) = S_2$

$\mathcal{F}$ is surjective. $\Leftarrow$ $S_2 \subseteq \mathcal{P}(B)$ $\exists S_1 \in \mathcal{P}(A)$, $\mathcal{F}(S_1) = S_2$

"$\Rightarrow$" DO IT YOURSELF!

Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a\,\rho\,a'$ if and only if $f(a)\,\sigma\,f(a')$.

(a) Prove that ρ is an equivalence relation on A.

(b) Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$. Prove that $\bar{f}$ is well-defined.

(a) Reflexive. $a\,\rho\,a \Leftrightarrow f(a)\,\sigma\,f(a)$ — Reflexive

eq. $\leftarrow \sigma_B$

f_A

$A/\rho = \{[x], [y], \ldots\}$

Symmetric:

if $f(a)\,\sigma\,f(a') \Rightarrow$ the $f(a')\,\sigma\,f(a)$ (given)

imply $a\,\rho\,a' \Rightarrow a'\,\rho\,a \longrightarrow \rho$ is symmetric

$B/\sigma = \{[f(x)], [f(y)], \ldots\}$

Transitive: complete yourself!

(b) $[a_1]_\rho = [a_2]_\rho \xrightarrow{\text{P.T}} [f(a_1)]_\sigma = [f(a_2)]_\sigma$

$a_1 \to b_1$, a_2, b_2

$a_1\,\rho\,a_2 \Rightarrow f(a_1)\,\sigma\,f(a_2)$

$\Rightarrow [f(a_1)]_\sigma = [f(a_2)]_\sigma$

$x \in [x]$

w.d. funct $\begin{cases} a_1 = a_2 \\ \Rightarrow f(a_1) = f(a_2) \end{cases}$

Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a \, \rho \, a'$ if and only if $f(a) \, \sigma \, f(a')$. Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$.

(c) Prove that $\bar{f}$ is injective.

(d) Prove or disprove: If f is a bijection, then so also is $\bar{f}$. $\rightarrow$ TRUE

(e) Prove or disprove: If $\bar{f}$ is a bijection, then so also is f. $\rightarrow$ FALSE

(c) if $[f(a_1)]_\sigma = [f(a_2)]_\sigma$ then $[a_1]_\rho = [a_2]_\rho$ [Prove this]

$f(a_1) \, \sigma \, f(a_2) \longleftrightarrow a_1 \, \rho \, a_2 \Rightarrow [a_1]_\rho = [a_2]_\rho$

(d) $\bar{f}$ is onto/surjective (remaining to be proved)

$\forall [f(a)]_\sigma \in B/\sigma \quad \exists [a]_\rho \in A/\rho \overset{\text{s.t.}}{\Longrightarrow} [a]_\rho \longmapsto [f(a)]_\sigma$

Since f is onto, $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$

$b \in [b]_\sigma$ $\quad \in [a]_\rho$

$\forall [b]_\sigma \in B/\rho \quad \exists [a]_\rho \in A/\rho$

$\bar{f}$ is also onto $\quad [a]_\rho \longmapsto [f(a)]_\sigma = [b]_\sigma$

$A = \{ x, y, z \}$

$B = \{ 1, 2 \}$

$\rho_A = \{ (x,x), (x,y), (y,x), (y,y), (z,z) \}$

(c) $f(x) = f(y) = 1$

$f(z) = 2$

$f: A \to B$

NOT bijective

$A/\rho = \{ [x]=[y], [z] \}$

$\sigma_B = \{ (1,1), (2,2) \}$

$B|_\sigma = \{ [1], [2] \}$

$\bar{f}: A/\rho \to B|_\sigma$

$\bar{f}([x]=[y]) = [1]$

$\bar{f}([z]) = [2]$

bijective

[FALSE]

one counterexample

[Genesis of rational numbers] Define a relation ρ on $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $(a, b)\ \rho\ (c, d)$ if and only if $ad = bc$. Prove that ρ is an equivalence relation. Argue that A/ρ is essentially the set $\mathbb{Q}$ of rational numbers.

Reflexive: $(a,b)\ \rho\ (a,b) \leftrightarrow ab = ba$ ✓

Symmetric: if $(a,b)\ \rho\ (c,d) \Rightarrow$ then $(c,d)\ \rho\ (a,b)$ ← This you prove!

$\hookrightarrow ad = bc \leftrightarrow cb = da$

$\forall, a, b$ $\exists (c, d)$

Transitive: P.T. $(a,b)\ \rho\ (c,d)$ and $(c,d)\ \rho\ (e,f)$ $\Rightarrow (a,b)\ \rho\ (e,f)$? $\frac{a}{b} = \frac{c}{d}$

$(1,2) \not\rho (2,3)$

$ad = bc$ $\quad$ $cf = de$ $\quad\rightarrow af = be$

$adcf = bcde$

$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \Rightarrow af = be$

$\boxed{\frac{a}{b} = \frac{c}{d}}$ ✗

$(1,2)\ \rho\ (2,4)$

$(1,2)\ \rho\ (3,6)$

$(1,2)\ \rho\ (4,8)$ . . . .

$[\frac{1}{3}]$ $[\frac{2}{3}]$ . . . . $[4/8] = [3/6] = [\frac{1}{2}] = \{(1,2), (2,4), (4,8), (3,6)$

Let ρ be a total order on A. We call ρ a *well-ordering* of A if every non-empty subset of A contains a least element. In this exercise, we plan to construct a well-ordering of $A = \mathbb{N} \times \mathbb{N}$.

(a) Define a relation ρ on A as $(a, b)\ \rho\ (c, d)$ if and only if $a \leq c$ or $b \leq d$. $\rightarrow \neg PO$

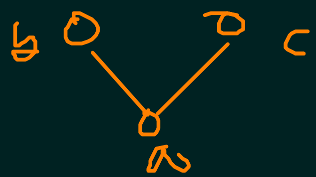
(b) Define a relation σ on A as $(a, b)\ \sigma\ (c, d)$ if and only if $a \leq c$ and $b \leq d$. $\rightarrow PO$, $\rightarrow \neg TO$

(c) Define a relation $\leq_L$ on A as $(a, b)\ \leq_L\ (c, d)$ if either (i) $a < c$ or $a = c$ and $b \leq d$. $\rightarrow PO$, TO, WO

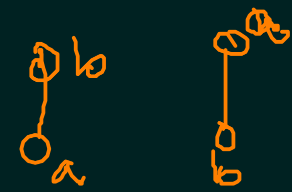
lexicographic sort (dictionary)

Prove or disprove: ρ, σ, $\leq_L$ is a well-ordering of A.

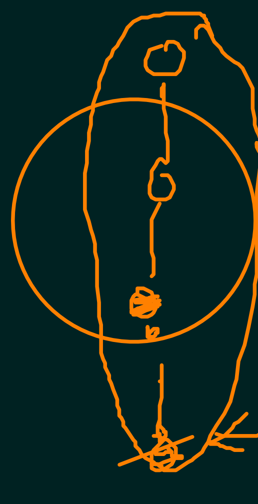
① Partial Order: R, AS, T



② Total order ρ_A: $\forall a, b \in A$, $(a,b) \in \rho$ or $(b,a) \in \rho$



! your Exercise !



well order

min ✓