Tutorial 7

(Un)Decidability, Rice's Theorem

1. Show that the set $\{\mathcal{M} \,|\, \mathcal{M}$ is a DFA not accepting any string with odd number of 1's$\}$ is decidable.
   Hint: For a DFA $\mathcal{M}$, the problem of whether or not $L(\mathcal{M}) = \emptyset$ is decidable.

2. Recall the definition of linear bounded automaton (LBA) and that the halting problem for LBA is decidable. Prove by diagonalisation that there exists a recursive set that is not accepted by any LBA.

3. True or False? It is decidable whether two given TMs accept the same set.

4. Show that $\{\mathcal{M}|\mathcal{M}$ is a TM that halts on all inputs of length less than $300\}$ is recursively enumerable but its complement is not.

5. Is the set $\{\mathcal{M} \mid \mathcal{L}(\mathcal{M})$ contains atmost 300 elements$\}$ *r.e.* ?

6. Show that none of the following languages or their complements are *r.e.*
   (a) $\mathsf{REG} = \{\mathcal{M} \mid \mathcal{L}(\mathcal{M})$ is a regular set$\}$.
   (b) $\mathsf{TOT} = \{\mathcal{M} \mid \mathcal{M}$ halts on all inputs$\}$.

7. Let
$$f(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}$$
   for any natural number x. Define $\mathsf{C}(x)$ as the sequence $x, f(x), f(f(x)), \ldots,$ which terminates if and when it hits 1. For example, if $x = 7$, then
$$\mathsf{C}(x) = (7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1).$$
   Computer tests have shown that $\mathsf{C}(x)$ hits 1 eventually for x ranging from 1 to 87×2^{60} (as of 2017). But, the question of whether $\mathsf{C}(x)$ ends at 1 for all $x \in \mathbb{N}$ is not proven. This is believed to be true and known as the Collatz conjecture. Suppose that MP were decidable by a Turing machine $\mathcal{K}$. Use $\mathcal{K}$ to describe a TM that is guaranteed to prove or disprove Collatz conjecture.

8. (a) Show that the language
$$\{(\mathcal{M}, \mathcal{N}) \mid \mathcal{M}, \mathcal{N} \text{ are Turing machines and } L(\mathcal{M}) \cap L(\mathcal{N}) = \emptyset\}$$
   is undecidable via reduction.
   (b) Prove the following extension of Rice's theorem (of which part (a) is a special case):
   Every non-trivial property of pairs of r.e. sets is undecidable.

More formally, let $\mathscr{P} : \{r.e.\text{ sets}\} \times \{r.e.\text{ sets}\} \to \{\top, \bot\}$ be a non-trivial property on pairs of *r.e.* sets. Then show that

$$T_{\mathscr{P}} = \{(\mathcal{M}, \mathcal{N}) \mid \mathcal{M} \text{ and } \mathcal{N} \text{ are TMs and } \mathscr{P}(L(\mathcal{M}), L(\mathcal{N})) = \top\}$$

is undecidable.