CS60005 Foundations of Computing Science
Tutorial 2

Set, Relation and Function

1. Let $A, B, C \in \mathcal{U}$ are three arbitrary sets such that, $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Prove that, $B = C$.

2. For a function, $f : A \to B$, define a function $\mathcal{F} : \mathcal{P}(A) \to \mathcal{P}(B)$ as $\mathcal{F}(S) = f(S)$ for all $S \subseteq A$. Prove that:

   (a) $\mathcal{F}$ is injective if and only if f is injective.

   (b) $\mathcal{F}$ is surjective if and only if f is surjective.

3. Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a \ \rho \ a'$ if and only if $f(a) \ \sigma \ f(a')$. Answer the following:

   (a) Prove that, ρ is an equivalence relation on A.

   (b) Prove or disprove: ρ defines a partial order over A.

   (c) Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$. Prove that, $\bar{f}$ is well-defined.

4. Let $f : A \to B$ be a function and σ an equivalence relation on B. Define a relation ρ on A as: $a \ \rho \ a'$ if and only if $f(a) \ \sigma \ f(a')$. Define a map $\bar{f} : A/\rho \to B/\sigma$ as $[a]_\rho \mapsto [f(a)]_\sigma$. Answer the following:

   (a) Prove that, $\bar{f}$ is injective.

   (b) Prove or disprove: If f is a bijection, then so also is $\bar{f}$.

   (c) Prove or disprove: If $\bar{f}$ is a bijection, then so also is f.

5. **[Genesis of rational numbers]** Define a relation ρ on $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $(a, b) \ \rho \ (c, d)$ if and only if $ad = bc$. Prove that ρ is an equivalence relation. Argue that A/ρ is essentially the set $\mathbb{Q}$ of rational numbers.

6. Let ρ be a total order on A. We call ρ a *well-ordering* of A if every non-empty subset of A contains a least element. In this exercise, we plan to construct a well-ordering of $A = \mathbb{N} \times \mathbb{N}$.

   (a) Define a relation ρ on A as $(a, b) \ \rho \ (c, d)$ if and only if $a \leq c$ or $b \leq d$.

   (b) Define a relation σ on A as $(a, b) \ \sigma \ (c, d)$ if and only if $a \leq c$ and $b \leq d$.

   (c) Define a relation $\leq_L$ on A as $(a, b) \ \leq_L \ (c, d)$ if either (i) $a < c$ or $a = c$ and $b \leq d$.

   Prove or disprove: ρ, σ, $\leq_L$ is a well-ordering of A.

7. Let A be the set of all functions $\mathbb{N}_0 \to \mathbb{R}^+$. Define a relation Θ on A as $f \ \Theta \ g$ if and only if $f = \Theta(g)$. Define a relation O on A as $f \ O \ g$ if and only if $f = O(g)$. Answer the following:

   (a) Prove that Θ is an equivalence relation.

   (b) Argue that O is not a partial order.

   Now, let us redefine the relation O on A/Θ as $[f] \ O \ [g]$ if and only if $f = O(g)$. Answer the following:

   (c) Establish that the relation O is well-defined.

   (d) Prove that O is a patial order on A/Θ.

   (e) Prove or disprove: O is a total order on A/Θ.

   (f) Prove or disprove: A/Θ is a lattice under O.