Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

Foundations of Computing Science (CS60005)		**Autumn Semester, 2021-2022**
Test - 1 [Marks: 30]	**Date: 08-Sep-2021 (Wednesday), 8:15am – 9:30am**	**Venue: Online**

[**Instructions**: ***There are FOUR questions. Answer ALL questions. Be brief and precise.***]

Q1. You are about to leave for university classes in the morning and discover you do not have your glasses. You know that the following *six* statements are true:

F_1 : *If my glasses are on the kitchen table, then I saw them at breakfast.*

F_2 : *I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.*

F_3 : *If I was reading the newspaper in the living room, then my glasses are on the coffee table.*

F_4 : *I did not see my glasses at breakfast.*

F_5 : *If I was reading my book in bed, then my glasses are on the bed table.*

F_6 : *If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.*

You task is to derive the answer to the following question logically – *"Where are the glasses?"*

Please frame logical arguments to formally deduce (applying logical inferencing) the answer to the above question. Present your solution as indicated in the following parts.

(a) Write all the propositions (that you have used) with English statements (meaning). **(1)**

(b) Build suitable propositional logic formula to encode each of the *six* statements $F_1 - F_6$ given above. **(3)**

(c) Use logical inferencing rules (or resolution-refutation principle) to completely derive the answer and conclude where do you find the glasses. **(3)**

Solution:

(a) We may use the following propositions.

p : My glasses are on the kitchen table.
q : I saw my glasses at breakfast.
r : I was reading the newspaper in the living room.
s : I was reading the newspaper in the kitchen.
t : My glasses are on the coffee table.
u : I was reading my book in bed.
v : My glasses are on the bed table.

(b) The proposition logic encodings are as follows.

$F_1 : \quad p \rightarrow q$
$F_2 : \quad r \vee s$
$F_3 : \quad r \rightarrow t$
$F_4 : \quad \neg q$
$F_5 : \quad u \rightarrow v$
$F_6 : \quad s \rightarrow p$

(c) The logical deduction procedure is given in the following.

$F_1 : \quad p \rightarrow q$
$F_4 : \quad \neg q$
$\therefore G_1 : \quad \neg p$ (Modus Tollens)

$F_6 : \quad s \rightarrow p$
$G_1 : \quad \neg p$
$\therefore G_2 : \quad \neg s$ (Modus Tollens)

$F_2 : \quad r \vee s$
$G_2 : \quad \neg s$
$\therefore G_3 : \quad r$ (Disjunctive Syllogism)

$F_3 : \quad r \rightarrow t$
$G_3 : \quad r$
$\therefore G : \quad t$ (Modus Ponens)

Conclusion: *The glasses are on the coffee table.*

Q2. Consider the following statements.

F_1 : *Tony and Mike are members of the Alpine club.*

F_2 : *Every member of the Alpine club is either a skier, or a mountain climber, or both.*

F_3 : *No mountain climber likes rain.*

F_4 : *All skier likes snow.*

F_5 : *Mike dislikes whatever Tony likes and likes whatever Tony dislikes.*

F_6 : *Tony likes rain and snow.*

Your tasks are to do the following:

(a) Write all the predicates (that you have used) with English statements (meaning). **(1)**

(b) Encode the above *six* statements $F_1 - F_6$ in predicate (first-order) logic. **(3)**

(c) Use resolution-refutation principle (logical deduction procedure) to prove that,
G : *"There is a member in the Alpine club who is a mountain climber, but not skier."* **(4)**

Solution:

(a) We may use the following predicates.

$member(x)$: x is a member of the Alpine club.

$climber(x)$: x is a mountain climber.

$skier(x)$: x is a skier.

$likes(x, y)$: x likes y.

(b) The predicate (first-order) logic encodings are as follows.

F_1 : $member(Tony) \wedge member(Mike)$

F_2 : $\forall x\big[member(x) \rightarrow (skier(x) \vee climber(x))\big]$

F_3 : $\forall x\big[climber(x) \rightarrow \neg like(x, Rain)\big]$

F_4 : $\forall x\big[skier(x) \rightarrow likes(x, Snow)\big]$

F_5 : $\forall x\Big[(likes(Tony, x) \rightarrow \neg likes(Mike, x)) \wedge (\neg likes(Tony, x) \rightarrow likes(Mike, x))\Big]$

F_6 : $likes(Tony, Rain) \wedge likes(Tony, Snow)$

(c) The goal statement can be encoded as follows.

$$G : \exists x\big[member(x) \wedge climber(x) \wedge \neg skier(x)\big] \quad \Longrightarrow \quad \neg G : \forall x\big[\neg member(x) \vee \neg climber(x) \vee skier(x)\big]$$

Now, $(F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge F_6 \rightarrow G)$ is valid $\Longrightarrow$ $(F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge F_6 \wedge \neg G)$ is unsatisfiable.

All the clauses formed from the above formula by eliminating $\forall$-quantifiers and *implications* are as follows.

C_{11} : $member(Tony)$

C_{12} : $member(Mike)$

C_2 : $\neg member(x) \vee skier(x) \vee climber(x)$

C_3 : $\neg climber(x) \vee \neg like(x, Rain)$

C_4 : $\neg skier(x) \vee likes(x, Snow)$

C_{51} : $\neg likes(Tony, x) \vee \neg likes(Mike, x)$

C_{52} : $likes(Tony, x) \vee likes(Mike, x)$

C_{61} : $likes(Tony, Rain)$

C_{62} : $likes(Tony, Snow)$

$C_{\neg G}$: $\neg member(x) \vee \neg climber(x) \vee skier(x)$

The resolution-refutation based deduction procedure is given in the following.

$$\frac{C_1 : \neg member(x) \vee skier(x) \vee climber(x) \qquad C_{\neg G} : \neg member(x) \vee \neg climber(x) \vee skier(x)}{\therefore D_1 : \neg member(x) \vee skier(x)}$$

$$\frac{C_{51} : \neg likes(Tony, x) \vee \neg likes(Mike, x) \qquad C_{62} : likes(Tony, Snow)}{\therefore D_2 : \neg likes(Mike, Snow)}$$

$$\frac{C_4 : \neg skier(x) \vee likes(x, Snow) \qquad D_2 : \neg likes(Mike, Snow)}{\therefore D_3 : \neg skier(Mike)}$$

$$\frac{D_1 : \neg member(x) \vee skier(x) \qquad D_3 : \neg skier(Mike)}{\therefore D_4 : \neg member(Mike)}$$

$$\frac{C_{12} : member(Mike) \qquad D_4 : \neg member(Mike)}{\therefore \quad \bot \quad (contradiction)}$$

Q3. A partial order ρ on a set A is called a total order (or a linear order) if for any two different $a, b \in A$ either $a\ \rho\ b$ or $b\ \rho\ a$. Which of the following relations $\rho,\ \sigma,\ \tau$ on $\mathbb{N}$ (the set of natural numbers) are partial orders and/or total orders? – *Provide proper reasoning / justification.*

[Hint: For each of the relations, $\rho,\ \sigma,\ \tau$ on $\mathbb{N}$, first determine whether the relation is a partial order, and if so, then determine whether it is a total order.]

(a) $a\ \rho\ b$ if and only if $a \leq b + 1701$. **(2)**

(b) $a\ \sigma\ b$ if and only if a divides b, i.e. $b = ax$, for some $x \in \mathbb{N}$. **(3)**

(c) $a\ \tau\ b$ if and only if either $u < v$, or $u = v$ and $x \leq y$, where $a = 2^u x$ and $b = 2^v y$ with x and y odd. **(3)**

Solution:

(a) Note that, $1\ \rho\ 2$ and $2\ \rho\ 1$, but $1 \neq 2$, i.e., ρ is not antisymmetric.

Hence, ρ is not a partial order on $\mathbb{N}$. So, obviously it can never be a total order.

(b) We have $a\ \sigma\ a$ (obvious as $a \in \mathbb{N}$ divides itself), indicating τ is reflexive.

Let $a\ \sigma\ b$ and $b\ \sigma\ a$. This implies that $b = ax$ (for some $x \in \mathbb{N}$) and $a = by$ (for some $y \in \mathbb{N}$). This is only possible when $x = y = 1$, implying $a = b$, i.e., σ is antisymmetric.

If $a\ \sigma\ b$ and $b\ \sigma\ c$, we have $b = ax$ and $c = by$ (for some $x, y \in \mathbb{N}$). So, we get, $c = by = (ax)y = az$, where $z = xy \in \mathbb{N}$, implying $a\ \sigma\ c$, i.e., σ is transitive too. Therefore, σ is a partial order on $\mathbb{N}$.

But σ is not a total order on $\mathbb{N}$, since neither $(2, 3)$ nor $(3, 2)$ belongs to σ.

(c) We have $a\ \tau\ a$ (obvious), indicating τ is reflexive.

Let $a = 2^u x$ and $b = 2^v y$ (with x, y odd) satisfy $a\ \tau\ b$ and $b\ \tau\ a$. We cannot have $u < v$ and $v \leq u$ simultaneously. So $u = v$. But then $x \leq y$ and $y \leq x$, implying $x = y$, i.e., $a = b$. So τ is anti-symmetric.

Now suppose $a\ \tau\ b$ and $b\ \tau\ c$, where $a = 2^u x$, $b = 2^v y$ and $c = 2^w z$ with x, y, z odd. We have $u \leq v$ and $v \leq w$, i.e., $u \leq w$. If $u < w$, then $a\ \tau\ c$. On the other hand, $u = w$ implies $u = v = w$. But then $x \leq y$ and $y \leq z$, so that $x \leq z$, i.e., $a\ \tau\ c$. So τ is a partial order on $\mathbb{N}$.

Finally, let $a = 2^u x$ and $b = 2^v y$ be two different integers. We then have either $u \neq v$ or $x \neq y$ (or both). If $u < v$, then $a\ \tau\ b$. If $u > v$, then $b\ \tau\ a$. If $u = v$, then $a\ \tau\ b$ or $b\ \tau\ a$ according as whether $x < y$ or $x > y$. Thus, τ is a total order on $\mathbb{N}$.

Q4. Prove or disprove the following *with proper reasoning / justification.*

(a) Let G be a multiplicative group in which $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.
Then, prove or disprove that G is Abelian. **(2)**

(b) Let R be a ring. Two elements $a, b \in R$ are called associates, denoted $a \sim b$, if $a = ub$ for some unit u of R.
Then, prove or disprove that $\sim$ is an equivalence relation on R. **(3)**

(c) Prove or disprove that the set of all finite subsets of $\mathbb{N}$ (the set of natural numbers) is countable. **(2)**

Solution:

(a) Let $a, b \in G$. By the given property $(a^{-1}b^{-1})^{-1} = (a^{-1})^{-1}(b^{-1})^{-1} = ab$.
Moreover, in any group $(a^{-1}b^{-1})^{-1} = (b^{-1})^{-1}(a^{-1})^{-1} = ba$.
Thus ab = ba. Therefore, G is Abelian. [Proved]

(b) **[Reflexive]** $a = 1 \times a$ for all $a \in R$.

[Symmetric] Let $a = ub$ for some unit u. Let $v \in R$ be the element with $uv = vu = 1$ in R. Then v is also a unit of R, and $b = va$.

[Transitive] Let $a = ub$ and $b = vc$ for some units u, v (i.e., $u^{-1}, v^{-1} \in R$). Then $a = (uv)c$. Moreover, $(v^{-1}u^{-1})(uv) = v^{-1}(u^{-1}u)v = v^{-1}v = e$, i.e., uv is also a unit in R.

Therefore, $\sim$ is an equivalence relation on R. [Proved]

(c) Let A denote the set of all finite subsets of $\mathbb{N}$. We write A as the disjoint union $A = \bigcup_{n \in \mathbb{N}_0} A_n$, where A_n comprises subsets of $\mathbb{N}$ of size n. $|A_0| = 1$. For $n \geq 1$ the set A_n can be identified with an (infinite) subset of $\mathbb{N}^n$ and so is countable. Since A is the union of countably many finite or countable sets, it is countable. [Proved]