$L \subseteq NL$

$L \overset{?}{=} NL$ → NL-Completeness defined in terms of log-space reductions.

Log space Reductions

$f : \{0,1\}^* \to \{0,1\}^*$ polynomially bounded fn.

[for any $x \in \{0,1\}^*$, $|f(x)| \leq |x|^c$ for some c.]

f is implicitly log-space computable if there exists a $\log(|x|)$-space DTM that given input (x, i), outputs $\boxed{f(x)|_i}$ provided $i \leq |f(x)|$.

i^{th} bit of $f(x)$.

$A, B \subseteq \{0,1\}^*$

A is log-space reducible to B, denoted $A \leq_l B$, if $\exists$ an implicitly log-space computable fn. $f : \{0,1\}^* \to \{0,1\}^*$ s.t. $x \in A$ iff $f(x) \in B$ $\forall x \in \{0,1\}^*$

Properties of $\leq_l$

1. $\leq_l$ transitive
2. $A \leq_l B$ & $B \in L$, then $A \in L$

1. $A \le_\ell B \le_\ell C$
$\quad f \qquad g$

M, N : log-space machines computing f, g

$A \le_\ell C$
$h = g \circ f$

$\boxed{K}$: log-space machine computing $g \circ f$.

K : on i/p (x, j), must o/p $g(f(x))|_j$

Simulates N. Whenever N needs to access i^{th} bit of $f(x)$, it pauses N, storing N's worktape contents. Then simulates M on x to obtain $f(x)|_i$ & then resumes N.

$O(\log|x|)$-space since N is a log-space machine

K should compute $g(f(x))_j$ without storing $f(x)$.

Storing indices $i \,/\, j$: $O(\log |f(x)|) \,/\, O(\log |g(f(x))|)$

$\equiv O(\log |x|)$ since f, g are polynomially bounded.

Storing N's worktape contents : $O(\log |x|)$

Total space requirements for K : $O(\log |x|)$.

NL-Completeness

A language A is NL-Complete if it is in NL & $\forall B \in NL$, $B \leq_{\ell} A$.

PATH = $\{ (G, s, t) : \exists$ a path in G from s to $t \}$

<u>Thm</u>: PATH is NL-Complete

<u>Proof</u>:

I PATH $\in$ NL

Maintain a counter k of no. of vertices visited $k \leftarrow 0$

Write down u on the worktape. (Initially $u = s$)

Non-deterministically choose a neighbour v of u

$u \leftarrow v$; $k{+}{+}$

If $u = t$ & $k \leq n \; (= |V(G)|)$, then accept.

; uses $O(\log n)$ space to store u, v & k

II PATH is NL-Complete

$A \in NL$, $A \leq_l PATH$.

$\Downarrow$

$\exists$ log-space NDTM M deciding A.

<u>Reduction</u>: $x \longmapsto (G_{M,x}, C_{start}, C_{accept})$

log-space computable.

configuration graph of M on i/p x.

C_{start}, C_{accept} : can be represented using $O(\log |x|)$ bits.

Consider the adjacency matrix of $G_{M,x}$

Every bit of $G_{M,x}$ can be obtained in $\log(|x|)$-space.

Bit at row C, column C' : $\begin{cases} 1 & \text{if } (C, C') \in E(G_{M,x}) \\ 0 & \text{o.w} \end{cases}$

can be determined by writing down $\boxed{C, C'}$ & simulating M's action in config C & checking if the next config is C'.

$O(\log |x|)$ space.