SPACE COMPLEXITY

Turing machine model

$S : \mathbb{N} \rightarrow \mathbb{N}$ $\qquad$ $S(n) \geqslant \log_2 n \quad \forall n$

DSPACE(S(n))

$L \in DSPACE(S(n))$ if $\exists$ a DTM M deciding L that uses at most $O(S(n))$ cells on its work tape on input of length n.

NSPACE $(S(n))$

$L \in NSPACE(S(n))$ if $\exists$ a NDTM M that decides L using atmost $O(S(n))$ cells on its work tape on input of length n.

Configuration Graph

TM M & string x.

$G_{M,x}$: directed graph

Vertices of $G_{M,x}$: configurations of M

$G_{M,x}$ contains a edge $C \to C'$ if M enters configuration C' from C after one application of transition function δ.

Assume that there is a unique accepting configuration.

$C_{start,x}$: starting configuration

C_{accept} : accepting configuration

M is an $S(n)$-space machine.

Assume that size of tape alphabet, no. of states are constant.

→ No. of bits required to represent one configuration : $c\,S(n)$

$c' \cdot S(n) + \underbrace{\log S(n)}_{\text{tapehead position}}$

→ No. of nodes $\leq 2^{c\,S(n)}$

→ There is an $O(S(n))$-sized CNF formula $\Phi_{M,x}(C, C')$ which is 1 iff $C \xrightarrow[M]{1} C'$.

$\mathrm{DTIME}(S(n)) \subseteq \mathrm{DSPACE}(S(n)) \subseteq \mathrm{NSPACE}(S(n)) \subseteq \mathrm{DTIME}(2^{O(S(n))})$

straightforward

In $S(n)$-time a TM can scan at most $S(n)$ cells in its tape.

Let $L \in \mathrm{NSPACE}(S(n))$.

M : $S(n)$-space NDTM deciding L.

Given x, construct $\underline{G_{M,x}}$

Takes $2^{O(S(n))}$ time

deterministically $\left\{\right.$ Check whether $\exists$ a path from $C_{start,x}$ to C_{accept}.

$\therefore L \in \mathrm{DTIME}(2^{O(S(n))})$

det. $2^{O(S(n))}$-time algorithm to check whether $x \in L$

$$PSPACE = \bigcup_{c>0} DSPACE(n^c)$$

$$NPSPACE = \bigcup_{c>0} NSPACE(n^c)$$

$$NSPACE(\log n) \subseteq DTIME(2^{O(\log n)}) = DTIME(2^{c \cdot \log n}) = DTIME(n^c) \subseteq P$$

$$L = DSPACE(\log n)$$

$$NL = NSPACE(\log n)$$

$$\begin{tikzcd} L \subseteq NL \subseteq P \arrow[rr, bend left, "\overset{?}{=}"] & \subseteq NP \subseteq & PSPACE \underset{=}{\boxed{\subseteq}} NPSPACE \subseteq EXP \subseteq EXPSPACE \end{tikzcd}$$

$$NSPACE(s(n)) \subseteq DTIME(2^{O(s(n))})$$

$$L \in NP$$

Given x, go through all potential certificates for x, verifying each one.
At any time only one certificate is stored on the tape $\Rightarrow L \in PSPACE$.