A language that is not r.e.

$$\neg HP = \{(M, x) \mid M \text{ does not halt on } x\}$$

If $\neg HP$ were r.e., then HP would be recursive (since we know HP is r.e.).

But HP is not recursive (i.e., undecidable)

So $\neg HP$ is not r.e.

$FIN = \{ M \mid L(M) \text{ is finite} \}$

Neither FIN nor $\neg$FIN is r.e.

$\rightarrow$ FIN is not r.e.

$$\neg HP \leq_m FIN$$

$$(M, x) \longmapsto N$$

N: on i/p y

- erases y
- simulates M on x
- accept if M halts

$(M, x) \in \neg HP \Rightarrow L(N) = \emptyset$ finite $\Rightarrow N \in FIN$

$(M, x) \notin \neg HP \Rightarrow L(N) = \Sigma^*$ infinite $\Rightarrow N \notin FIN$

Since $\neg$HP is not r.e., neither is FIN.

→ $\neg$FIN is not r.e.

$$\neg HP \leq_m \neg FIN$$

$$(M, x) \longmapsto N$$

N : on i/p y

- saves y on a separate track
- simulate M on i/p x for $|y|$ steps
- accept if M does not halt (within $|y|$ steps)

$(M, x) \in \neg HP$ i.e., M does not halt on x

$\Rightarrow$ N will accept all strings $\Rightarrow L(N) = \Sigma^*$

$\Rightarrow N \in \neg FIN$

$(M, x) \notin \neg HP$ i.e., M halts on x, say, in n steps.

$\Rightarrow$ N will accept strings y with $|y| < n$

$\Rightarrow$ $L(N) = \Sigma^{<n}$ finite $\Rightarrow N \notin \neg FIN$

$\neg HP$ is not r.e $\Rightarrow$ $\neg FIN$ is not r.e

RICE'S THEOREM

✓ Problems about r.e. sets (an r.e. set is finitely described by a TM)

Given TM M

- → Is $L(M) = \Sigma^*$? monotone
- → Is $L(M) = \phi$? non-monotone
- → Is $L(M)$ regular? non-monotone
- → Is $L(M)$ a CFL? non-monotone
- → Does $L(M)$ contain string y? monotone

not r.e.

✗ Problems about TMs

Give TM M

- → Does M have atleast 100 states?
- → Is M equivalent to a smaller TM?

Property

$$P : \{\text{r.e. subsets of } \Sigma^*\} \longrightarrow \{T, F\}$$

$[F \leq T]$

P is non-trivial if it is neither universally true nor universally false

P is monotone if $A \subseteq B \Rightarrow P(A) \leq P(B)$ $\forall A, B$.

P is non-monotone if it is not monotone.
($\exists A, B$ s.t. $A \subseteq B$ but $P(A) = T$ & $P(B) = F$)

Rice's Theorem Part I : Every non-trivial property of r.e. sets is undecidable

Proof : Assume w.l.g. that $P(\phi) = F$.

Let A be an r.e. set with $P(A) = T$.

Let K be a TM accepting A.

$$T_P = \{ M \mid P(L(M)) = T \}$$

P is undecidable $\Leftrightarrow$ T_P is not recursive.

$$HP \leq_m T_P$$

$$(M, x) \longmapsto N$$

N : on i/p y
- save y on a separate track
- run M on x
- if M halts, run K on i/p y
- accept if K accepts

$(M,x) \in HP \Rightarrow L(N) = A$ ie, $N \in T_P$

$(M,x) \notin HP \Rightarrow L(N) = \phi$ ie, $N \notin T_P$

$\therefore$ T_P (or P) is undecidable