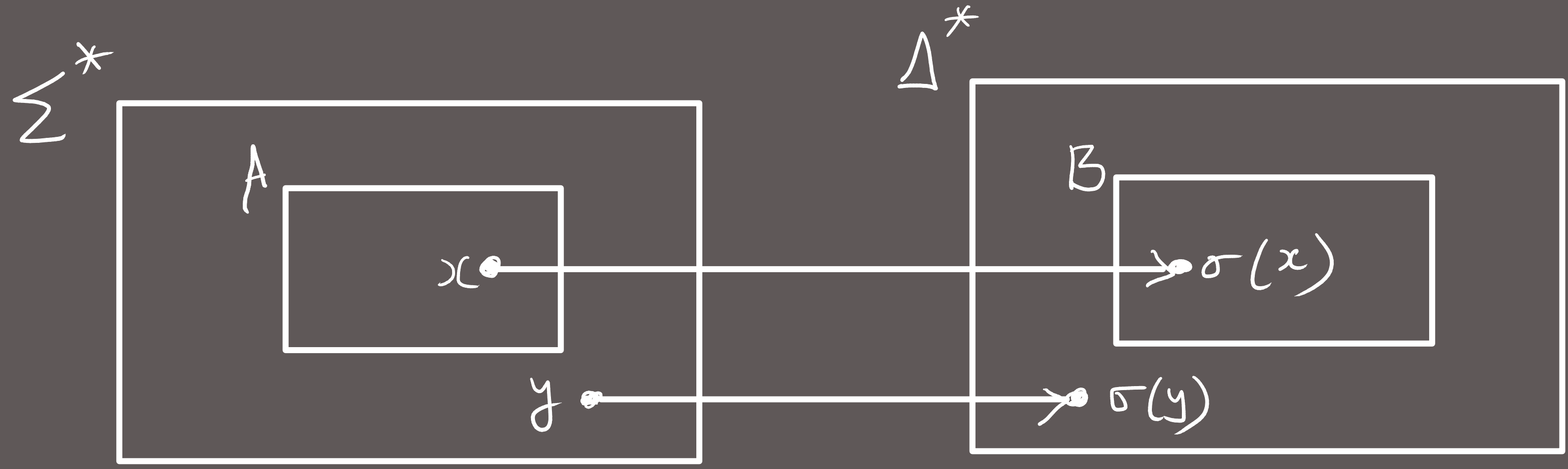
Reduction

$A \subseteq \Sigma^*$, $B \subseteq \Delta^*$

A many-one reduction from A to B is a computable function $\sigma : \Sigma^* \to \Delta^*$ s.t.

$$\forall x \in \Sigma^*, \quad x \in A \iff \sigma(x) \in B$$



σ : computable by a total Turing machine

$\boxed{A \leq_m B}$ if $\exists$ a many-one reduction from A to B.

$\leq_m$ is transitive

$A \leq_m B$ & $B \leq_m C$

σ : reduction $A \to B$ τ : reduction $B \to C$

$\tau \circ \sigma$: reduction fn. $A \to C$

Since τ, σ are total computable, so is $\tau \circ \sigma$

$\forall x$ $x \in A \iff \sigma(x) \in B \iff \tau(\sigma(x)) \in C$

$\iff \tau \circ \sigma(x) \in C$

Σ Δ

$A \leq_m B$ & A is undecidable

What can you say about B?

Suppose that B is decidable.

$x \in \Sigma^*$. Let σ be the reduction from A to B.

Use the decider for B to check whether $\sigma(x) \in B$, which tells us whether $x \in A$. $\Rightarrow$ A is decidable!!!

Thm If $A \leq_m B$ and

(i) B is decidable, then so is A

(ii) A is undecidable, then so is B

(iii) B is r.e., then so is A

(iv) A is not r.e., then B is not r.e.

Example

$$L = \{ M \mid L(M) = \Sigma^* \}$$

$HP \leq_m L$

(M, x) : instance of HP.

σ should be defined s.t. $(M, x) \in HP$ iff $\sigma(M, x) \in L$

$\sigma : (M, x) \longmapsto N$

N : on input y

- erases the input
- simulates M on x
- accept if M halts

M, x : hardcoded in N's description.

Description of N can be written down in finite time.

σ, given M, x, writes down description of N.

$(M, x) \in HP \Rightarrow M$ halts on $x \Rightarrow L(N) = \Sigma^*$

$\Rightarrow N \in L$

$(M, x) \notin HP \Rightarrow M$ does not halt on $x \Rightarrow L(N) = \emptyset$

$\Rightarrow N \notin L$

$\therefore$ $HP \leq_m L$. Since HP is undecidable, so is L.

Same reduction works when

undecidable

$$L = \{ M \mid L(M) \neq \emptyset \}$$

$$L = \{ M \mid \varepsilon \in L(M) \}$$