UNIVERSAL TURING MACHINES

$\mathcal{U}$: i/p is description of a TM

A TM can be encoded using any alphabet Σ

Encoding to describe a TM (with $\{0,1\}$)

$$M = (Q, \Sigma, \Gamma, \delta, \vdash, \sqcup, s, t, r)$$

Encoding for states in Q : $0, 0^2, 0^3, \ldots, 0^{|Q|}$

Encoding for Γ : $\underbrace{0, 0^2, 0^3, \ldots, 0^{|\Sigma|}}_{\text{symbols of } \Sigma}, 0^{|\Sigma|+1}, \ldots, 0^{|\Gamma|}$

Encode M as

$$0^{|Q|} 1 0^{|\Gamma|} 1 \underbrace{0^{u}}_{\vdash} 1 \underbrace{0^{v}}_{\sqcup} 1 0^{s} 1 0^{t} 1 0^{r} 1 \underbrace{\ldots}_{\text{encoding for } \delta}$$

Encoding δ : $\delta(p,a) = (q,b,R) \rightsquigarrow 0^{p} 1 0^{a} 1 0^{q} 1 0^{b} 1 \boxed{1}$ moving Right

encoding of L : 0

Consider a Turing machine U which takes as input (M, x) where M is a TM & x is a string.

$$L(U) = \{(M, x) \mid M \text{ accepts } x\}$$

U just simulates the machine M on i/p x.

- could use 3 tracks
  - Track 1 contains description of M
  - track 2 simulates the tape contents of M (initially x)
  - track 3 simulates transitions of M.

UNDECIDABILITY

A language (or property) is undecidable if it is not decidable i.e., not recursive.

Halting Problem

$$P_{HP}(M,x) = \begin{cases} T & \text{if } (M,x) \in HP \\ F & \text{o.w} \end{cases}$$

$$HP = \{ (M,x) \mid M \text{ halts on } x \}$$

Membership Problem

$$MP = \{ (M,x) \mid M \text{ accepts } x \}$$

MP, HP are r.e. but not recursive

Diagonalisation [Cantor]

There does not exist an onto function

$$f : \mathbb{N} \to 2^{\mathbb{N}}$$

Thm HP is undecidable

Proof: Suppose that HP is decidable. $\exists$ a total TM $\mathcal{U}$ s.t. $\mathcal{U}$ decides HP

That is, on i/p (M, x), $\mathcal{U}$ accepts if M halts; rejects if M loops.

A: accept

R: reject

Enumeration of all strings

Enumeration of all TMs

$\mathcal{U}$	ϵ	0	1	00	01	10	11	$\cdots$
M_ϵ	A	R	R	A	A	R	R	
M_0	A	R	R	R	A	A	A	
M_1	R	A	R	R	A	R	R	
M_{00}	A	R	R	A	R	R	A	
M_{01}								
$\vdots$								

M_y : TM with encoding y

K : on input x

- construct description of M_x
- run $\mathcal{U}$ on i/p M_x, x
- accept if $\mathcal{U}$ rejects; reject if $\mathcal{U}$ accepts

$x \in L(K)$ iff $M_x, x \notin L(U)$

K's behaviour differs from that of M_x for all x at least on one i/p.

$K \notin \{M_\epsilon, M_0, M_1, \ldots\}$

But $M_\epsilon, M_0, M_1, \ldots$ is an enumeration of all TMs.

Contradicts our assumption that HP is decidable.