Context Free Languages:

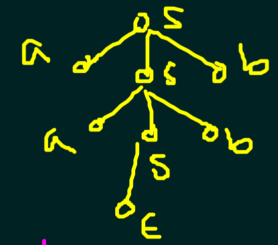
Grammar

$G = (N, \Sigma, P, S)$

Derivations

$S \xrightarrow[G]{*} x \equiv S \xrightarrow[G]{n} x \ (n \geq 0)$

Parse Tree



Regular Grammars

↳ left-linear : $A \to Bw$
↳ Right-linear : $A \to wB$

} Equivalent NFA/DFA

$[A] \xrightarrow{w_1} [w_2 \dots w_k B]$

→ Start Symbol (initial state)
→ $[\epsilon]$ symbol (final state)

Left Linear → Reverse Gram. → NFA → Rev edges → RL Gram.

SUMMARY

Pumping Lemma:

Let α be any CFL

$\exists$ pumping Constant n, s.t. $|z| \geq n$ and $z \in \alpha$

↳ then we may write:

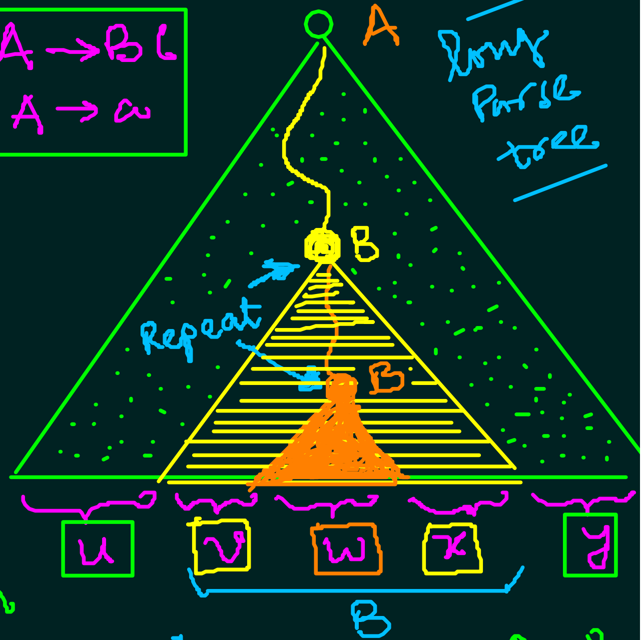
① $|uy| \geq 1$ ② $|vwx| \leq n$, and

③ $uv^iwx^iy \in L \ (\forall i \geq 0)$

$[CNF]_G$: $A \to BC$, $A \to a$

$A \xrightarrow[G]{*} uBy$
$B \xrightarrow[G]{*} vBx$
$B \xrightarrow[G]{*} w$

Derive



$A \xrightarrow[G]{*} uBy \xrightarrow[G]{*} uvBxy \xrightarrow[G]{*} uv^2Bx^2y$

$uv^iwx^iy \xleftarrow[G]{*} \dots \xleftarrow[G]{*} uv^3Bx^3y \xleftarrow{*}_G$

$|uy| \geq 1$
So $|vwx| \leq n$

Similarly, $A \xrightarrow[G]{*} uBy \xrightarrow[G]{*} uwy \equiv uv^0wx^0y$

↳ To contradict, need to prove existance of one i where $uv^iwx^iy \notin L$ ✓

Closure Properties of CFL:

$G_1 = (N_1, \Sigma, P_1, S_1)$ → α_1 and $G_2 = (N_2, \Sigma, P_2, S_2)$ → α_2

↳ $G(\alpha_1 \cup \alpha_2) = (N_1 \cup N_2 \cup \{S\}, \Sigma, P_1 \cup P_2 \cup \{S \to S_1 | S_2\}, S)$

↳ $G(\alpha_1 \alpha_2) = (N_1 \cup N_2 \cup \{S\}, \Sigma, P_1 \cup P_2 \cup \{S \to S_1S_2\}, S)$

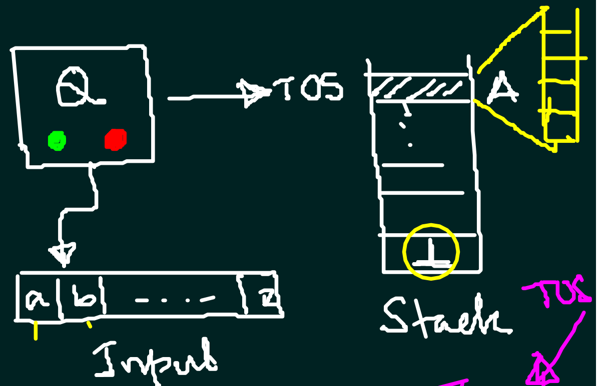
↳ $G(\alpha_1^*) = (N_1 \cup \{S\}, \Sigma, P_1 \cup \{S \to S_1S | \epsilon\}, S)$

↳ NOT under intersection (CF)

Ex: $\{a^ib^ic^j \mid i,j \geq 1\} \cap \{a^ib^jc^j \mid i,j \geq 1\}$
(CF)
$\Rightarrow \{a^ib^ic^i \mid i \geq 1\}$
$(\neg CF)$

(NOT under complement)

Push-Down Automaton (PDA): $M = (Q, \Sigma, \Gamma, \perp, \delta, s, F)$



Q – set of states; Σ – input symbols; Γ – stack symbols; s – start state; F – set of final states

$\perp \in \Gamma \rightarrow$ bottom of stack

$\boxed{\delta} \rightarrow$ transition fn

↳ Input: $p \in Q$ (state)

$a \in \Sigma \cup \{\epsilon\}$, $A \in \Gamma$

$\longmapsto$ Output: $q \in Q$ (any state)

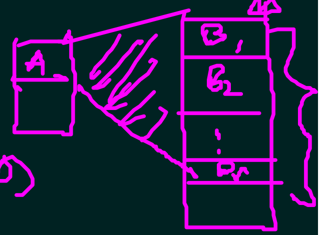
$A / B_1 B_2 \ldots B_k \in \Gamma^*$ $(k \geq 0)$

$k = 0$: A/ϵ

$k = 1$, $B_1 = A$: A/A

$$\delta \subseteq \left[(Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma), (Q \times \Gamma^*) \right]$$

$$p \xrightarrow{\boxed{a},\, A / B_1 B_2 \ldots B_k} q$$



▷ Acceptance Criteria:

↳ final state $\dashrightarrow$ $f \in F$ (stack $\rightarrow \gamma \in \Gamma^*$) ✓

↳ empty stack $\dashrightarrow$ $F = \emptyset$, TOS $\rightarrow$ X

} write

▷ Points to Note:

– Each transition requires $A \in$ stack(Γ) ✓

– ϵ – transitions are allowed.

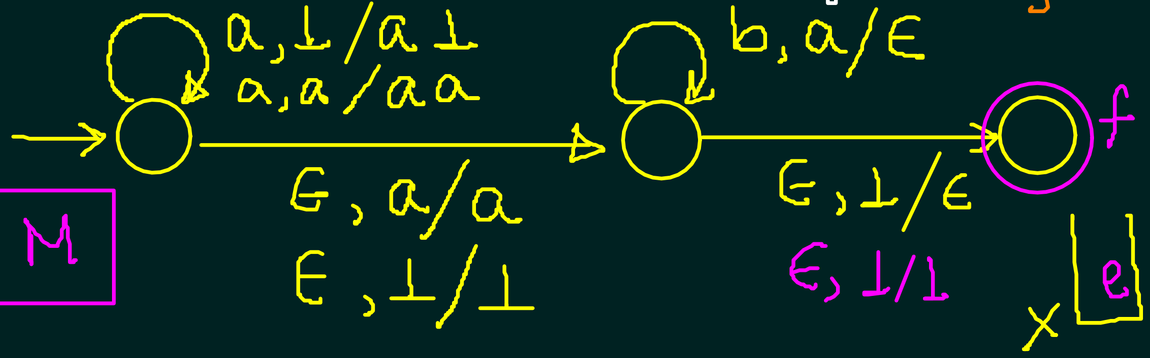
– PDAs are non-deterministic by def.

"aaabbaabab"

"Acceptance is justified when you fully read the string"

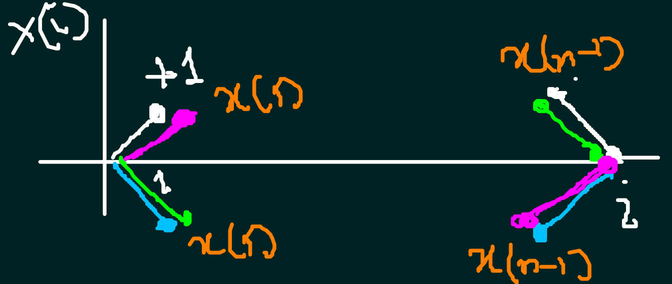
Examples: ① $\{a^n b^n \mid n \geq 0\}$ $\Sigma = \{a, b\}$ $\Gamma = \{\perp, a\}$

$\rightarrow \underbrace{aaa \ldots a}_{n} \mid \underbrace{bbb \ldots b}_{n}$



H/W: $\{a^m b^n \mid m \geq n \geq 0\}$

② $\{w \in \{a,b\}^* \mid \#a(w) = \#b(w)\}$ aababbab ✓



Case-1
$w_1 = a$
$w_n = b$

Case 2
$w_1 = b$
$w_n = a$

$\begin{cases} X(0) = 0 \\ X(n) = 0 \end{cases}$

$w_1\ w_2\ w_3 \ldots w_n$

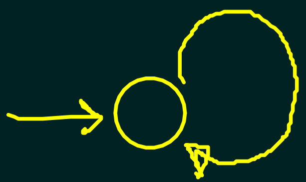
$\boxed{X(i)} = \#a \text{ read} - \#b \text{ read}$

Case3
$w_1 = a$
$w_n = a$

Case4
$w_1 = b$
$w_n = b$

$E \rightarrow \epsilon \mid aEb \mid bEa \mid EE$

$+_a \quad -_b$



$a, + / ++$
$a, - / \epsilon$
$a, \perp / +\perp$
$b, + / \epsilon$
$b, - / --$
$b, \perp / -\perp$

empty stack

$\boxed{\epsilon, \perp / \epsilon}$

a b b a ε

TOS → ⊕ —a→ push +, —b→ pop

TOS → ⊖ —a→ pop, —b→ push −

③ $L = \{ww \mid w \in \{a,b\}^*\}$ is NOT CF. ✓ $\quad \underbrace{a^n b^n}\, \underbrace{a^n b^n} \quad (n = k+l+1)$

~L is CF → ~L = ① strings of odd length
② If even length,

(1) scan $\boxed{k}$ symbols and push '✓' $\quad \boxed{2n}$

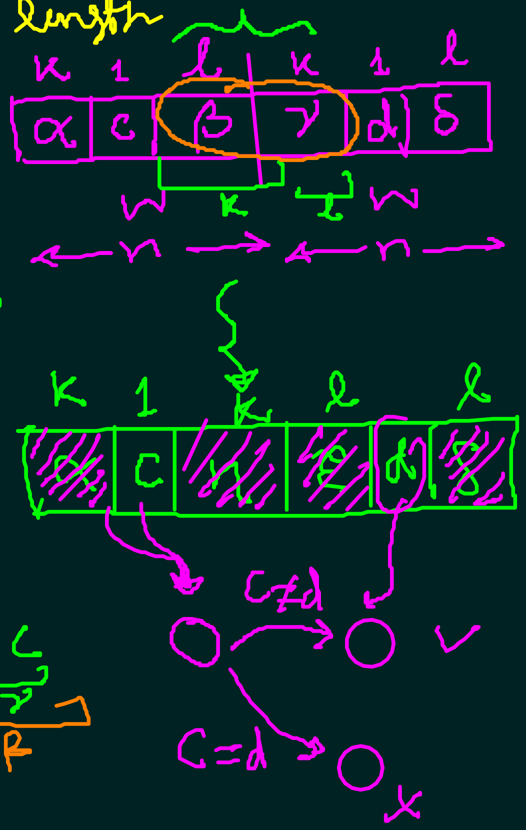
② remember c $\quad$ ✓ $c \neq d$

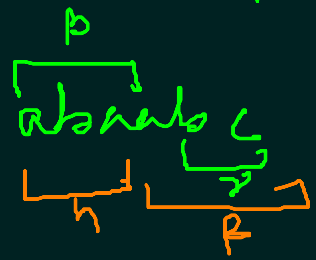
(3) scan k symbols and pop '✓' $\quad \underbrace{\boxed{\beta + \gamma}}_{k+l} = n + \varepsilon$

(4) scan l symbols and push '✓'

(5) read d and check c = d → reject
d ≠ c → proceed

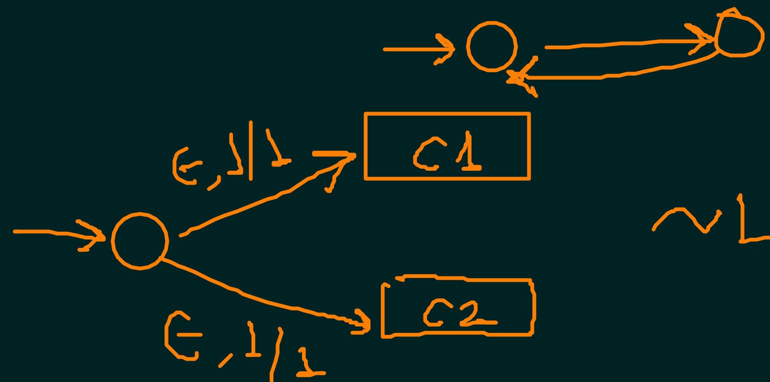
(6) scan l symbols and pop '✓'





▷ Strings of odd length

abacb $\qquad a^k b^k$



~L

→ Properties of PDA & CFG relation

$M = (Q, \Sigma, \Gamma, \perp, \delta, s, F)$ $\qquad ((p, a, A), (q, \gamma)) \in \delta \qquad \gamma \in \Gamma^*$

"finite"

$$\delta \subseteq (Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$$

$\left(\begin{array}{l} F \subseteq Q \\ s \in Q \end{array}\right)$

▷ <u>Configuration:</u> $C \in Q \times \Sigma^* \times \Gamma^*$

↳ (Initial) $\rightarrow (s, x, \perp)$ $\qquad (q, y, \gamma)$

(final) $\rightarrow$ accept by f.s. $\rightarrow (q, \epsilon, \gamma)$, $q \in F$, $\gamma \in \Gamma^*$

$\qquad\rightarrow$ accept by empty stack $\rightarrow (q, \epsilon, \epsilon)$

$((p, a, A), (q, \gamma)) \in \delta$

↳ $(p, ay, A\beta) \xmapsto[M]{1} (q, y, \gamma\beta)$

$((p, \epsilon, A), (q, \gamma)) \in \delta$

↳ $(p, y, A\beta) \xmapsto[M]{1} (q, y, \gamma\beta)$

one step change of conf

$C \xmapsto[M]{0} D \iff C = D$ $\qquad C \xmapsto[M]{*} D$

$C \xmapsto[M]{n+1} D \iff \exists$ a conf E s.t. $C \xmapsto[M]{n} E$ and $E \xmapsto[M]{1} D$, $n \geq 0$

$M \to \alpha(M)$ "empty stack" ✓

$$\alpha_1(M) = \{ x \in \Sigma^* \mid (s, x, \bot) \xrightarrow[M]{*} (q, \epsilon, \epsilon) \text{ some } q \in Q \}$$

$$\alpha_2(M) = \{ x \in \Sigma^* \mid (s, x, \bot) \xrightarrow[M]{*} (q, \epsilon, \gamma),\ q \in F,\ \gamma \in \Gamma^* \}$$

final state

x

$\boxed{\epsilon x \in \Sigma^*}$

$$M = (Q, \Sigma, \underline{\Gamma}, \bot, \delta, s, F)$$ → empty stack → final state

$$M' = (Q', \Sigma, \Gamma', \amalg, \delta', s', \{t\})$$ → accepts both by empty stack / final state

$Q' = Q \cup \{s'\} \cup \{t\}$ $\qquad \Gamma' = \Gamma \cup \{\amalg\}$ $\quad (\amalg \notin \Gamma)$

M'

(1) $((s', \epsilon, \amalg), (s, \bot\amalg)) \in \delta'$ ✓

(2) all δ's are there in δ' ✓

(3) M accepts by empty stack ✓ (q, ϵ, ϵ) ✓ | M accepts by final state (q, ϵ, γ)

$((q, \epsilon, \amalg), (t, \epsilon)) \in \delta'$

fs, empty stack

$((q, \epsilon, A), (t, A)) \in \delta'$

$((t, \epsilon, A), (t, \epsilon)) \in \delta'$ ✓

$A \in \Gamma \cup \{\amalg\}$

$\boxed{\alpha(M) = \alpha(M')}$

Equivalence of PDA with CFG: ① CGF → PDA $\boxed{M}$

$\boxed{A \to \gamma}$ $(A \in N,\ \gamma \in (\Sigma \cup N)^*)$

$G = (N, \Sigma, P, S) \longrightarrow (\{*\}, \Sigma, \Gamma, S, \delta, *, \emptyset)$

$\Gamma = \Sigma \cup N$

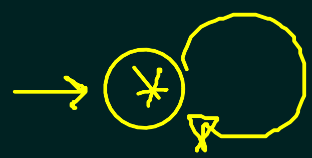
$((*, \epsilon, A), (*, \gamma)) \in \delta$ ✓

$\forall a \in \Sigma,\ ((*, a, a), (*, \epsilon)) \in \delta$ ✓

$\mathcal{L}(M) = \mathcal{L}(G)$

→ M simulates the leftmost deriv. of G

$E \to \epsilon \mid aEb \mid bEa \mid EE$



$\epsilon, E/\epsilon$ ✓

$\epsilon, E/aEb$

$\epsilon, E/bEa$

✓ $\epsilon, E/EE$

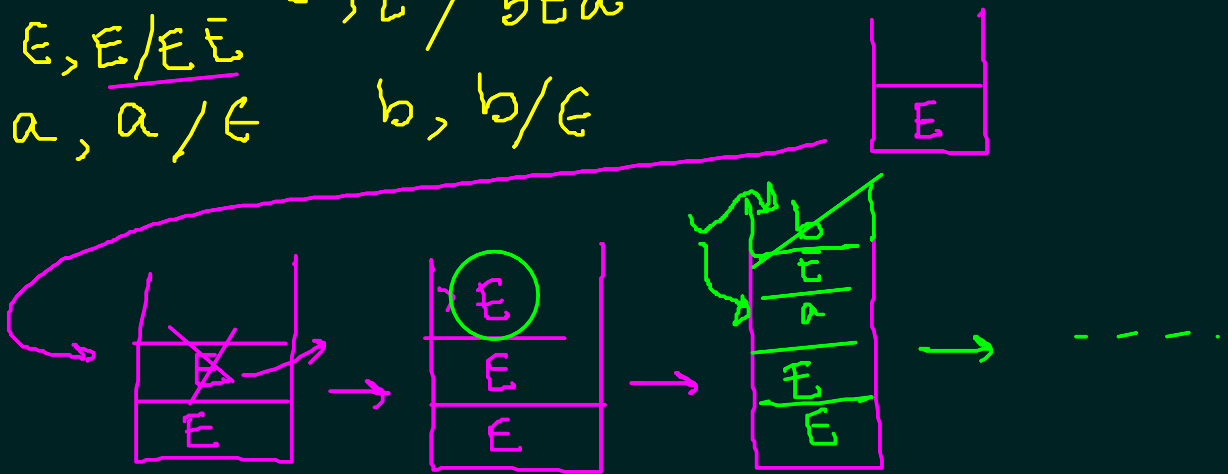
$a, a/\epsilon$

$b, b/\epsilon$

empty stack Accept

$\boxed{baabba}$

$E \to EE \to EEE \to bEaEE \xrightarrow{\epsilon} baEE \to baaEbE \to baabE \to baabEa \to baabba$



leftmost derivation

$\boxed{\text{PDA} \to \text{CFG}}$

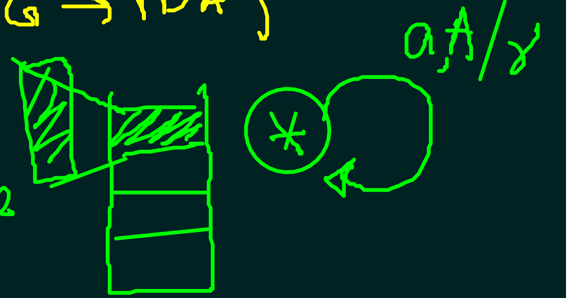
$M \to G$

① $M \longmapsto M'$ with only one state and empty stack accept.

② $\underline{M'} \longmapsto G$ (reverse process of (CFG → PDA))

② $((*, a, A), (*, \gamma)) \in \delta$

$\hookrightarrow \boxed{A \to a\gamma}$ production Rule



① Can M' general enough?

$\boxed{M}$ (accepts by final state as well as empty stack) $\longmapsto \boxed{M'}$ $(\{*\}, \Sigma, \Gamma', \bot', \delta', *, \phi)$

$(Q, \Sigma, \Gamma, \bot, \delta, s, \{t\})$

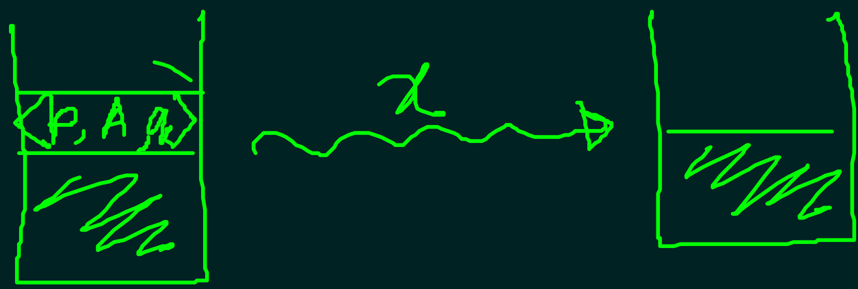
empty stack

$s \ \bot \longrightarrow t$

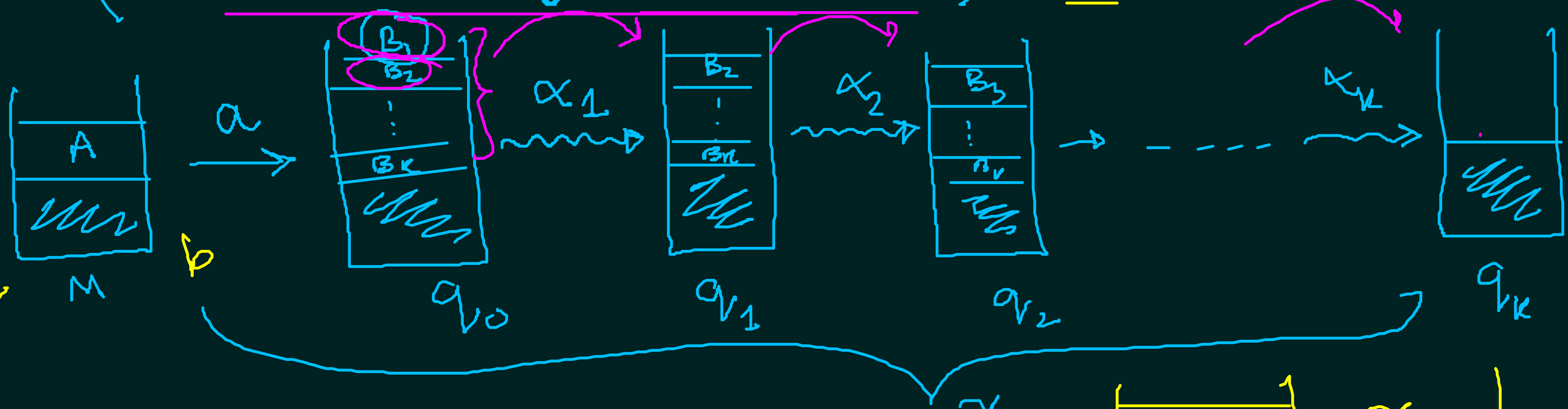
$\Gamma' = (Q \times \Gamma \times Q)$

$\langle p \ A \ q \rangle$

$\boxed{\bot' = \langle s \ \bot \ t \rangle}$



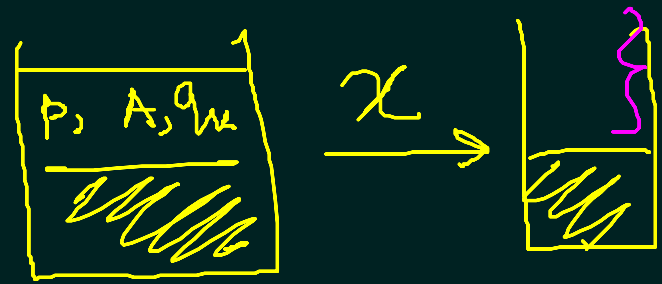
$M \quad ((p, a, A), (q_0, B_1 B_2 \dots B_k)) \in \delta$



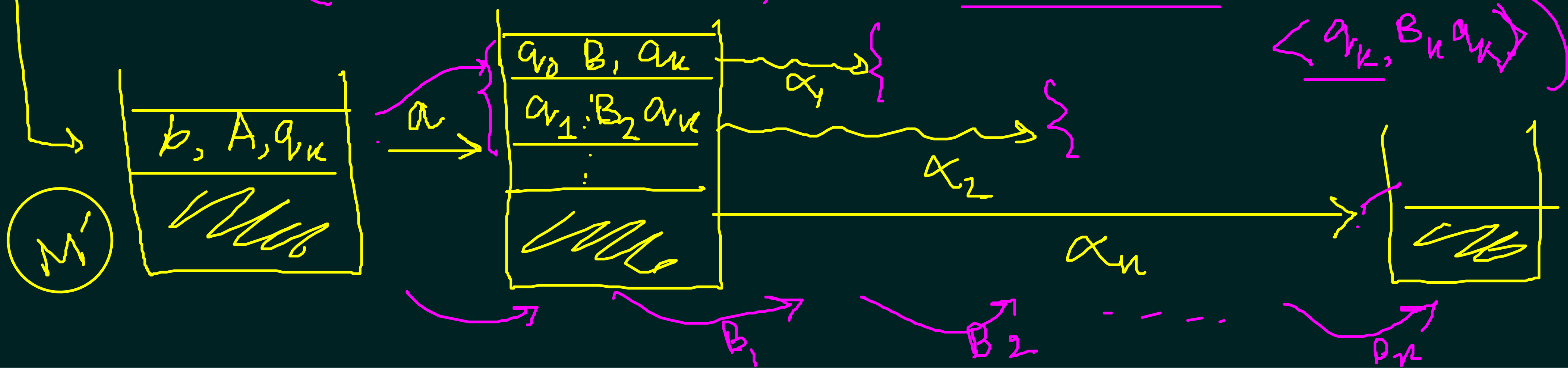
M' guessing
$(q_0, q_1 \dots q_k)$

$x = a\,\alpha_1 \alpha_2 \dots \alpha_k$

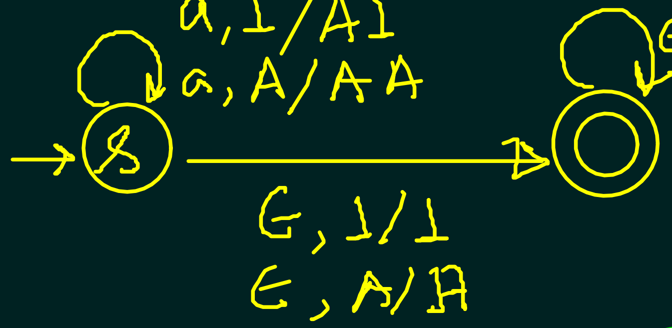
$\boxed{A \to a\alpha's}$



$\delta' \to M'$ $\quad (*, a, \langle p, A, q_k\rangle), \; (*, \langle q_0, B_1, q_1\rangle \langle q_1, B_2, q_2\rangle \dots \langle q_k, B_k, q_k\rangle)$



$\boxed{\{a^n b^n \mid n \geq 0\}}$



$(s, a, \bot), (s, A\bot) \in \delta$

$(s, a, A), (s, AA) \in \delta$

$(s, \epsilon, \bot), (t, \bot) \in \delta$

$(s, \epsilon, A), (t, A) \in \delta$

$(t, b, A), (t, \epsilon) \in \delta$

$((t, \epsilon, \bot), (t, \epsilon) \in \delta$

PDA

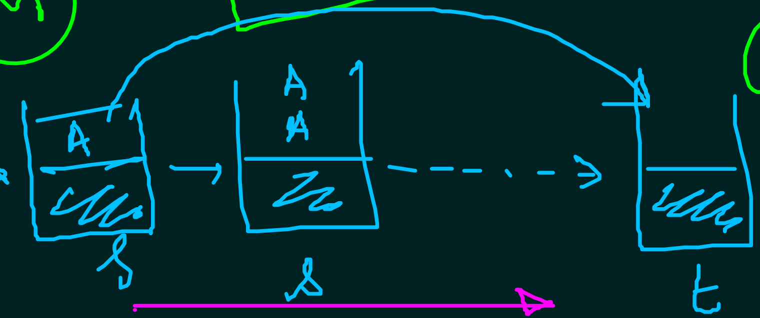
M

$\langle s \bot t \rangle$

$\langle s A t \rangle$

$\langle t \bot t \rangle$

$\langle t A t \rangle$



X

$(*, a, \langle s \bot t \rangle), (*, \underbrace{\langle s A t \rangle}_{B_1} \underbrace{\langle t \bot t \rangle}_{B_2})$

$(*, a, \langle s A t \rangle, (*, \langle s A t \rangle \langle t A t \rangle)$

$\langle s A s \rangle$ X

$\langle s \bot s \rangle$ X

$\langle t \bot s \rangle$ X

$\langle t A s \rangle$ X

aabb

Induction: $\mathcal{L}(M) = \mathcal{L}(M)$