Sizes of Sets

Countable — Uncountable

A - Set $\quad |A| \to$ finite $< \infty$

$|A| = \infty$ (not clear)

$\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ infinite counting process

$\exists$ sets where you cannot exhaust all elements $\to$ uncountable

▷ A, B - sets, $|A| \leq |B|$ if $\exists$ an injective map $f: A \to B$

↳ f produces an embedding of A to B.

Ex: ① $A \subseteq B$ $\left\{ \begin{array}{l} \iota : A \to B \\ a \mapsto a \end{array} \right.$ $\quad |A| \leq |B|$

② $|\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{R}|$ $\qquad |\mathbb{N}_{odd}| \leq |\mathbb{N}|$

$|\mathbb{N}_{even}| \leq |\mathbb{N}|$

③ $|\mathbb{Z}| \leq |\mathbb{N}|$

$0 \mapsto 1$
$1 \mapsto 2$
$-1 \mapsto 3$
$2 \mapsto 4$
$-2 \mapsto 5 \dots$

$n \mapsto 2n$
$-n \mapsto 2n+1$

bijective

$0 \mapsto 1$
$1 \mapsto 3$
$-1 \mapsto 5$
$2 \to 7$
$-2 \to 9$

$n \to 4n-1$
$-n \to 4n+1$

injective, but not onto

$|\mathbb{Z}| \leq |\mathbb{N}|$

$|\mathbb{Z}| = |\mathbb{N}|$

$|A| = |B|$ if $|A| \leq |B|$ and $|B| \leq |A|$

$\equiv$ $\exists$ injective maps s.t $f : A \to B$ and $g : B \to A$

A, B - equinumerous

Ex: $|\mathbb{Z}| = |\mathbb{N}|$

Theorem: [Cantor - Shröder - Bernstein]

$|A| = |B|$ if and only if $\exists$ a bijection $h : A \to B$

$f^{-1} : B \to A$

Countable Sets:

Th: Let A be any infinite set. then $|\mathbb{N}| \leq |A|$

Proof: $f : \mathbb{N} \to A$ injective (To prove)

$a_1 \in A$, $f(1) = a_1$ .... $f(i) = a_i$ $\forall i = 1, 2, \ldots, n.$

$a_{n+1} \neq a_i$ (any i) $f(n+1) = a_{n+1}$ Induction

Corollary: $|\mathbb{N}|$ is the smallest infinity $\checkmark$ $= \aleph_0$ (aleph-not)

def: 'A' is countable if $|A| < \infty$, or $|A| = |\mathbb{N}|$

A is Countable if $|A| < \infty$, or $|A| = |\mathbb{N}|$ ✓
(infinite)

$\iff \exists$ an injective map $f : A \to \mathbb{N}$

$\iff \exists$ a bijective map $\boxed{h : \mathbb{N} \to A}$ (CSB Th.)

$$A = \{\underset{a_1}{h(1)}, \underset{a_2}{h(2)}, \dots, \underset{a_n}{h(n)}, \dots\}$$ ← infinite counting process

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Theorem: 1 Any subset of Countable Set is countable.

Proof $A \subseteq B$ (Countable)

$\therefore$ A Countable.

$i : A \to B$ injective

$$\left.\begin{array}{l} |A| \leq |B| = |\mathbb{N}| \\ \text{and } |\mathbb{N}| \leq |A| \end{array}\right\} |A| = |\mathbb{N}|$$

2 Union of two Countable Sets is Countable.

Proof: $A = \{a_1, a_2, \dots, a_n, \dots\}$

$B = \{b_1, b_2, \dots, b_m, \dots\}$

$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$

$A \cap B \neq \emptyset$ we do not list.

$A \cup B$ Countable.

③ Let $k \in \mathbb{N}$, and $A_1, A_2, \dots, A_k$ are countable. Then $\bigcup_{i=1}^{k} A_i$ is also countable.

Proof: [Induction] $k=1$ ✓ $k=n$ $\left(\bigcup_{i=1}^{n} A_i\right) = B$ is count

$$(\underline{B} \cup A_{\underline{n+1}}) = \bigcup_{i=1}^{n+1} A_i \leftarrow \text{countable.}$$

④ The union of countably many countable sets is countable.

$\{A_n, \ n \in \mathbb{N}\} \to$ a collection of countable set

$$A_n = \{a_{n,1}, a_{n,2}, a_{n,3}, a_{n,4}, \dots\}$$

a_{ij} → $i = 1, 2, 3$; $j = 1, 2, \dots$

Proof:

$$\begin{array}{c|cccccc}
A_1 \ 1 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \dots \\
A_2 \ 2 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \dots \\
3 & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \dots \\
4 & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \dots \\
5 & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \dots
\end{array}$$

$[a_{11},$ ← $i+j=2$

$[a_{12}, a_{21}$ ← $i+j=3$

$[a_{13}, a_{22}, a_{31}$ ← $i+j=4$

⋮

5 A, B — countable then $A \times B \Rightarrow$ countable.

Proof: $\forall a \in A$, $B_a = \{(a,b) \mid b \in B\}$

$B_{a_1} = \{(a_1,b_1), (a_1,b_2), (a_1,b_3) \ldots\}$

$f : B \to B_a$
each $b \mapsto (a,b)$ } bijection

$\Rightarrow B_a$ countable (each)

$A \times B = \bigcup_{a \in A} B_a \Rightarrow$ Countable

$\begin{cases} a_1b_1 & a_1b_2 & \ldots \\ a_2b_1 & a_2b_2 & \ldots \\ a_3b_1 & a_3b_2 & \ldots \end{cases}$

~~6~~ Corollary $\mathbb{Q}$ is countable

Proof: $\mathbb{Q} = \left\{ a/b \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{N},\ \gcd(a,b) = 1 \right\}$

$\times \left[\begin{matrix} (1, 2) \\ (2, 4) \end{matrix} \right.$

$\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{N}$ → countable → $\mathbb{Q}$ = countable

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$\aleph_0 + \aleph_0 = \aleph_0$

$k \in \mathbb{N}$ $\quad k\aleph_0 = \aleph_0$

$\aleph_0 \times \aleph_0 = \aleph_0$

$\aleph_0^k \ (k \in \mathbb{N}) = \aleph_0$

$|\mathbb{N}| = |A|$ ← A is countable

(Smallest infinity)

$|\mathbb{N}_{odd}| = |\mathbb{N}_{even}| = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{P}| = |\mathbb{Q}|$
$= \aleph_0$

▷ $|\mathbb{R}|$ is not countable.

↳ $[0,1) = \{x \in \mathbb{R} \mid 0 \le x < 1\}$

not countable

proper fractions

$0.a_1 a_2 a_3 \ldots\ldots 2$

Ex:
$\pi - 3 = 0.\cancel{1}4159265 35\ldots$

$3/8 = 0.3\cancel{7}5 0000 \ldots$ ✓

$= 0.37\cancel{4}9999 \ldots$

$f(1) = 0.\boxed{a_{11}}\, a_{12}\, a_{13}\, a_{14} \ldots$

$f(2) = 0.a_{21}\, \boxed{a_{22}}\, a_{23}\, a_{24} \ldots$

$f(3) = 0.a_{31}\, a_{32}\, \boxed{a_{33}}\, a_{34} \ldots$

$f(4) = 0.a_{41}\, a_{42}\, a_{43}\, \boxed{a_{44}} \ldots$

$\vdots$

$f(n)$ k k k k^3

$\vdots$

$b = 0.b_1 b_2 b_3 b_4 \ldots$ ✓

A is uncountable set

$\Leftrightarrow$ $f: A \to \mathbb{N}$ we cannot find injective map

$\boxed{|\mathbb{N}| \le |A|}$ ✓ $|A| \le |\mathbb{N}|$

$\Leftrightarrow$ No map $\mathbb{N} \to A$ is bijective

$\Downarrow$ ✓

Let $f: \mathbb{N} \to [0,1)$ be "injective".

We prove that f is not onto.

$f(1), f(2), \ldots \ldots$ 0.aaaaa…

$$b_i = \begin{cases} 1 & \text{if } a_{ii} = 2 \\ 2 & \text{if } a_{ii} \ne 2 \end{cases}$$

$b \ne f(i)$

Diagonalization proof

$k = |\Sigma| \ge 2$ $\to k^{\aleph_0} > \aleph_0$

[**Theorem**]: There cannot be any bijection from A to $\mathcal{P}(A) = 2^A$

Proof: $f : A \to \mathcal{P}(A)$ form injective function

$a \mapsto \{a\} \quad \Rightarrow \quad |A| \leq |\mathcal{P}(A)|$

n	$f(n)$	elements: 1	2	3	4	5	6	…
1	$\emptyset$	$\boxed{\cancel{0}}$	0	0	0	0	0	0 …
2	$\{2,3,4,6\}$	0	$\cancel{1}$	1	1	0	1	0 - -
3	$\{1,9,12,16\}$	1	0	$\boxed{\cancel{0}}$	0	0	0	- - - 1 0 0 1 - - .
4	$\{1,2,4,9,10\}$	1	1	0	$\boxed{\cancel{1}}$	0	-	- - -
5	$\{3,6,7,11,12,13\}$	0	0	1	0	$\boxed{\cancel{0}}$	1	1 - - - .
⋮								
n^*	$\{1,3,5,\ldots\}$	1	0	1	0	1	-	.

$|A| < |\mathcal{P}(A)|$

$|\mathbb{R}| = \mathfrak{c}$ (continuum)

$\mathfrak{c} \geq [0,1) > \aleph_0$

$|\mathbb{R}_{\geq 0}| = |[0,1)|$

$f : \mathbb{R}_{\geq 0} \to [0,1)$

$x \mapsto \dfrac{x}{x+1}$ is bijection

$|\mathbb{R}_{\geq 0}| \leq |\mathbb{R}|$

$g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ an injective map $\Rightarrow |\mathbb{R}| \leq |\mathbb{R}_{\geq 0}|$

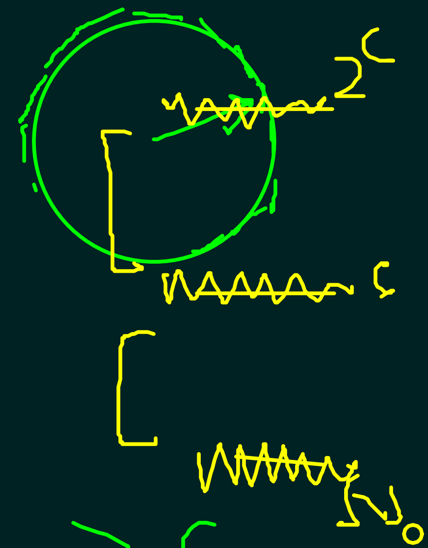
$$x \longmapsto \begin{cases} \dfrac{x}{x+1}, & x \geq 0 \\ \dfrac{-x}{-x+1} + 1, & x < 0 \end{cases}$$

$|\mathbb{R}| = |\mathbb{R}_{\geq 0}|$

$= |[0,1)| = c$

$2^{\aleph_0} = c$

$$\begin{cases} 0.\boxed{1}10 \ldots \\ 0.0\boxed{1}0 \ldots \\ \end{cases}$$



$\aleph_0 < c$

infinity (countable)

bigger infinity (uncoun

$|2^{\mathbb{R}}| = 2^c > c$

∞ countable number of (uncountable) infinities $\Big\}$ $\cdots\cdots$ $2^{2^{2^c}} > 2^{2^c} > 2^c > c$

c and $\aleph_0 \to$ what in between?

(ZF) + axiom of choice $\to$ × prove

→ Not all problems can be solved by computer.

↳ There are uncountably many Problems

↳ Computers can only prove countably many Problems

▷ Alphabet : $\Sigma = \{0,1\}$, $\{0-9\}$, $\{I, II, \ldots V, L, \ldots\}$

Σ, Γ, Δ ASCII $\{a-z, A-Z\}$

▷ String : over Σ is a finite ordered sequence of symb. from Σ

$\Sigma = \{0,1\} \longrightarrow \left\{ \begin{matrix} 0\,0\,1\,1\,0\,0 \\ 0\,1\,0\,1\,1\,1 \\ 1\,1\,1\,0\,1\,1 \end{matrix} \right\}$ $\Sigma^* =$ set of all finite strings over Σ

→ $\boxed{\Sigma^* \text{ is countable}}$; $\Sigma^* = \bigcup_{l \geq 0} \Sigma^l$ $|\Sigma^l| = |\Sigma|^l$ ↳ finite countable

∴ Σ^* is countable union of countable sets

↳ $\mathcal{P}(\Sigma^*)$ is uncountable ✓ (Proposition)

▷ Language: $\mathcal{L} \subseteq \Sigma^*$ ✓ $\{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$

Prime $= \{ \underbrace{2, 3, 5, 7, 11, 13, \ldots}_{} \}$ ✓ $\in$

▷ Problems:

Given a language $\mathcal{L}$ over Σ a string $\alpha \in \Sigma^*$, deciding whether $\alpha \in \mathcal{L}$

Repr

p

primality test

↳ Representing language:

$\{ (r, c, val)$ ( ) ( ) {

{english, $\boxed{\{ a \in \mathbb{N} \mid a \text{ is prime} \}}$

C - language $\equiv$ grammar int i

description / grammars $\in \Gamma^*$

↳ countably many languages have finite description

Problems solved by computer } $\subseteq$ Language having finite descriptions $\boxed{\mathcal{P}(\Sigma^*)}$

countably

Unsolvable Problems → (1) Halting Problem

Cantor ✓
Gödel
Turing

"Dangerous Knowledge"

$p \rightarrow M$ → "Yes" if p halts
→ "NO" if p loops

before Yes: while(1)

before No: return

$p \rightarrow M'$

M' loops when ~~p~~ M' halts

M' stops when ~~p~~ M' loops

Contr.

n	i_1	i_2	i_3	i_4 - -
1	$\boxed{\cancel{0}}$	1	1	0
2	1	$\cancel{1}$	1	0
3	1	0	$\boxed{\cancel{1}}$	0
	-	-	-	-

Answer

New U_1: 1 0 1

\# Solvable Problem $= \aleph_0 \rightarrow$ (Countable)

\# Unsolvable Problems $= 2^{\aleph_0} \rightarrow$ (uncountable)