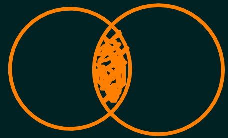
SUMMARY

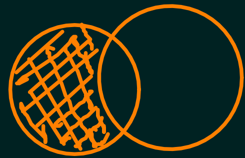
▷ **Set** : $A = \{x \mid x \text{ has a property}\}$ Membership : $x \in A$.

Subset : $B \subseteq A \rightarrow |B| \leq |A|$ Power Set : $\mathcal{P}(A)$ = set of all subsets of A

$A \cup B$



$A \cap B$



$A - B \equiv A \cap \overline{B}$

$A \triangle B$

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

↳ Set operations

Cartesian Product : $A \times B = \{(a,b) \mid a \in A, b \in B\}$

▷ **Relation** : $\rho \subseteq A \times B$ $|A \times B| = |B \times A| = |A||B|$ but $A \times B \neq B \times A$

↳ Reflexive : ρ_A if $\forall a \in A$, $(a,a) \in \rho_A$

↳ Symmetric : ρ_A if $\forall a,b \in A$, s.t. $(a,b) \in \rho_A \rightarrow (b,a) \in \rho_A$

↳ Transitive : ρ_A if $\forall a,b,c \in A$ s.t. $(a,b) \in \rho_A$ & $(b,c) \in \rho_A \rightarrow (a,c) \in \rho_A$

↳ Antisymmetric : ρ_A if $\forall a,b \in A$ s.t. $(a,b) \in \rho_A$ & $(b,a) \in \rho_A \rightarrow a = b$

Equivalence Relation

poset

▷ Equivalence Relation ρ over A induces a partition of A. → each partition forms equivalence class.

$[y] = \{x \mid (x,y) \in \rho\}$ A/ρ

① $x \in [x]$ ② $(x,y) \in \rho \longleftrightarrow [x] = [y]$, ③ $[x] = [y]$ or $[x] \cap [y] = \phi$

Poset → Hasse Diagram

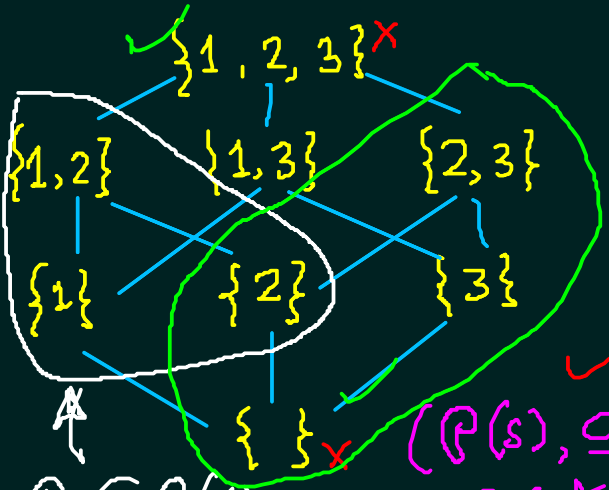
$(\mathcal{P}(A), \subseteq)$

Maximal	Minimal	Least	Greatest
{1,2,3}	{ }	{ }	{1,2,3}

can be multiple unique or none

Total order

Poset + $\forall a,b \in A$ $(a,b) \in \rho$



$B \subseteq P(S)$

$(P(S), \subseteq)$ poset

$\boxed{\text{at most one glb and lub}}$

Lower Bound

(A, ρ) poset and $B \subseteq A$

$x \in A$ is LB of B if $\forall b \in B,\ (x, b) \in \rho$

Ex: $\{\ \} \in B$ and $\in A$

Greatest lower Bound (glb)

Upper Bound

(A, ρ) poset and $B \subseteq A$

$y \in A$ is UB of B if $\forall b \in B,\ (b, y) \in \rho$

Ex: $\{1, 2\} \in B$ and $\in A$

$\{1, 2, 3\} \in A$. $\notin B$

Least upper Bound (lub)

▷ **Lattice**: (A, ρ) poset is a lattice if every pair of elements $a, b \in A$, lub$\{a, b\}$ and glb$\{a, b\}$ exists in A.

↳ Complete when all subsets of elements has glb and lub.

Ex: ① Poset (N, ρ), $\rho = \{(x, y) \mid x \leq y,\ x, y \in N\} \longrightarrow$ lattice.

lub$\{x, y\} = \max\{x, y\}$ and glb$\{x, y\} = \min\{x, y\}$

② Poset $(P(S), \rho)$, $\rho = \{(A, B) \mid A \subseteq B,\ A, B \in P(S)\} \rightarrow$ lattice
(Complete)

lub$\{A, B\} = A \cup B$ and glb$\{A, B\} = A \cap B$

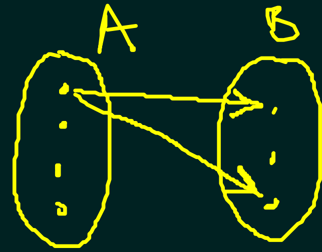
③ Poset $(\mathbb{Z}^+, \rho)$, $\rho = \{(x, y) \mid x \text{ div } y,\ x, y \in \mathbb{Z}^+\} \rightarrow$ lattice

lub$\{x, y\} = \text{LCM}(x, y)$ and glb$\{x, y\} = \text{GCD}(x, y)$.

▷ $Q \subset P(S)$ $\emptyset \notin Q$ then poset $(Q, \subseteq)$ ✗

Functions: $f : A \to B$

$\left\{\begin{array}{l}(a, b) \in f \\ (a, c) \in f\end{array}\right\} \Rightarrow b = c$

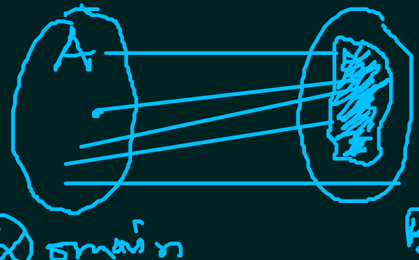


Relation ($\neg$ function)

$f(x) = \lceil x \rceil$, $x \in \mathbb{R}$

$g(y) = \lfloor y \rfloor$, $y \in \mathbb{R}$

Codomain



Domain

$f(A) \subseteq B$

Range

① $A_1, A_2 \subseteq A$ $\quad f(A_1) \subseteq f(A_2)$ if $A_1 \subseteq A_2$

② $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

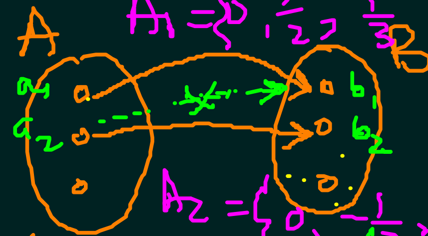
$x \in f(A_1 \cup A_2) \xrightarrow{\subseteq} x \in [f(A_1) \cup f(A_2)]$

③ $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$A_1 = \{0, \frac{1}{2}, \frac{1}{3}, \dots$



$A_2 = \{0, -\frac{1}{2}, -\frac{1}{3}, \dots\}$

Types: One-to-one (injective)

Onto (surjective)

$f(a_1) = b_1$

$f(a_2) = b_2$ if $b_1 = b_2$

then $a_1 = a_2$

$|A| \leq |B|$ ✓

$|A| \geq |B|$

$f(A) = B$

$\forall b \in B \;\; \exists a \in A$

s.t. $f(a) = b$

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 2x + 1$

$f(x) = x^2 + x \to \times \to$ one-to-one

$f^{-1}(A_1 \cap A_2) = f^{-1}(A_1) \cap f^{-1}(A$

Onto: $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x+1$ ✓ $f(x) = x^3+1$

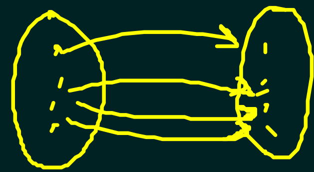
[Not] $f(x) = x^2$

$y = -4$

$x = 2i / -2i$

⇒ Bijective = One-to one and Onto

$|A| = |B|$



$\boxed{\text{Ex (H/W): } f(x) = \sqrt{x}}$

$4 \to +2,\ -2$ $f: \mathbb{R} \to \mathbb{R}$

$f: A \times A \to B$ → $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$+2 \oplus -3 = -1$

example → $f(x,y) = f(y,x)$ → Commutative

$f(f(x,y), z) = f(x, f(y,z))$ ← associative

Identity: $a + 0 = 0 + a = a$

$f(x,a) = f(a,x) = a$ $\forall a \in A$

(x is id.)

unique x_1, x_2

$f(x_1, x_2) = x_2$

$f(x_1, x_2) = x_1$

$g: A \to B$

$f: B \to C$

$f \circ g(x) = f(g(x))$

Identity function $f(a) = a$

$1_A: A \to A$

$1_B: B \to B$

$\boxed{f \circ 1_A = f = 1_B \circ f}$

▷ $f\circ g(x) \neq g\circ f(x)$

$ \quad \| \qquad\qquad\quad \|$

$(x+1)^2 \neq x^2+1$

$\begin{cases} f(x)=x^2 \\ g(x)=x+1 \end{cases}$ $\quad f:\mathbb{R}\to\mathbb{R}$, $\; g:\mathbb{R}\to\mathbb{R}$

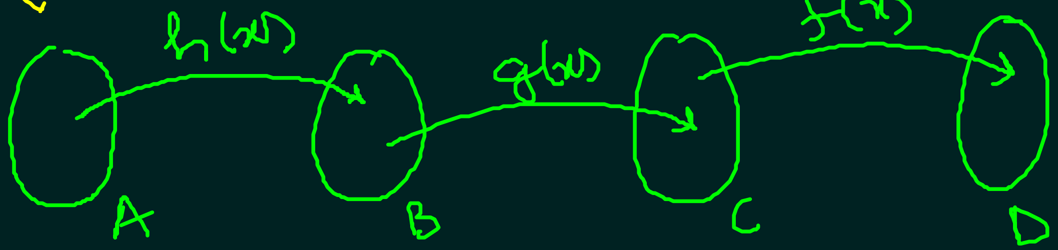
▷ $f\circ(g\circ h)(x) = (f\circ g)\circ h(x)$ ✓

$f: C\to D$

$g: B\to C$

$h: A\to B$

(Associative)



$f\circ(g(h(x))) = f(g(h(x))) = (f\circ g)(h(x)) = (f\circ g)\circ h(x)$

▷ $f: A\to B$ ✓, $g: B\to C$ one-to-one $\Big\}$ $g\circ f: A\to C$ is one-to-one

(i) $f(a_1) = f(a_2) \longleftarrow g(\underbrace{f(a_1)}_{b_1}) = g(\underbrace{f(a_2)}_{b_2}) \Longrightarrow a_1 = a_2$?

(ii) $a_1 = a_2$

▷ $f: A\to B$, $g: B\to C$ onto $\Big\}$ $g\circ f: A\to C$ onto ✓

$\forall c\in C \quad \exists b\in B$

s.t $g(b)=c$

$f: A \to B$, $g: B \to C$ } if $g \circ f: A \to C$ is one to one

$\to$ $\therefore$ f is one-to one, but g need not ?

$g(f(a_1)) = g(f(a_2)) \longrightarrow f(a_1) = f(a_2)$

f is not one-to-one $\to$ $\exists x_1, x_2 \in A \Rightarrow f(x_1) = f(x_2)$

$\to g(f(x_1)) = g(f(x_2))$

HW

Ex

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if $g \circ f: A \to C$ is onto.

g is onto, but f need not.

HW ?

Not onto

Ex: $g, f: \mathbb{Z} \to \mathbb{Z}$ $\quad f(x) = 2x$

$g(x) = \left\lfloor \frac{x}{2} \right\rfloor$

$g \circ f(x) = x \to$ onto

Inverse: $f: A \to B$

$f^{-1}: B \to A$

$f_L^{-1} \circ f = 1_A$ and $f \circ f_R^{-1} = 1_B$

f^{-1}

$f_1^{-1} = f_1^{-1} \circ 1_B$

$= f_1^{-1} \circ (f \circ f_2^{-1}) = 1_A \circ f_2^{-1} = f_2^{-1}$

f_1^{-1}, f_2^{-1} assume

$f^{-1} = f_1^{-1} = f_2^{-1}$

▷ $f: A \to B$ is invertible iff f is bijective.

"$\Rightarrow$" $f^{-1} \circ f = 1_A \to$ inject $\to f$ inj

$f \circ f^{-1} = 1_B \to$ onto $\to f$ surj

'$\Leftarrow$' $y \in B$ has one & only one preimage $x \in A$

$f^{-1}: B \to A \qquad f^{-1}(y) = x$

$f^{-1} \circ f(x) = f^{-1}(y) = x \qquad f \circ f^{-1}(y) = f(x) = y$

$\Rightarrow f^{-1} \circ f = 1_A \quad , \quad f \circ f^{-1} = 1_B \to$ invertible

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▷ $f: A \to B$
if $g: B \to C$ } invertible, then $g \circ f: A \to C$ is invertible

$\Downarrow$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$f, g \to$ biject $\to g \circ f$ bij

$(f^{-1} \circ g^{-1}) \circ (g \circ f) = 1_A$ - - - -

$f \circ f^{-1} = 1_B$
$f^{-1} \circ f = 1_A$