$\boxed{\text{Set}}$: well-defined collection of distinct objects

↳ Ex: $S = \{4, 9, 16, \ldots, 81, 100\} = \{x^2 \mid x \text{ is integer and } 1 < x \leq 10\}$

Membership: $25 \in S$, $72 \notin S$

Cardinality: $|S| = 9$

Finite Set: $|S| < \infty$

Infinite Set: $|S| = \infty$?

$T = \{2^y \mid y \text{ is int and } y \geq 0\}$

Subset ⟶ **Proper Subset** $A \subset B$

$A \subseteq B \equiv \forall x [x \in A \to x \in B]$ + $|A| < |B|$

$|A| \leq |B|$

Equal Sets

$A = B \equiv A \subseteq B$ and $B \subseteq A$

$|A| = |B|$

Null Set: NO elements $A = \phi$, $|A| = 0 = |\phi|$

$\phi \neq \{0\}$, $\phi \neq \{\phi\}$ $|\{0\}| = |\{\phi\}| = 1$

Power set: $A = \{1, 2, 3\}$ $\mathcal{P}(A) = 2^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$|2^A| = 8$

Generally, $|\mathcal{P}(A)| = 2^{|A|}$

$\sum_{k=0}^{n} {}^nC_k = 2^n$ if $|A| = n$.

$A \in \mathcal{P}(A)$

$\{1, 2\} = \{2, 1\}$

Properties of Set:

$A \subseteq B, \; B \subseteq C \longrightarrow A \subseteq C$

$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ $\to \mathbb{Z}^+ = \mathbb{Z}_{>0}$, $\to \mathbb{Z}^- = \mathbb{Z}_{<0}$

$\mathbb{Q} = \{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z}^+ \}$ $\to \mathbb{Q}^+$, $\to \mathbb{Q}^-$

$\mathbb{Q}^*$, $\mathbb{R}^*$ ↳ non-zero

$\mathbb{R}$ = set of reals

$\mathbb{C}$ = Set of complex $= \{ a+ib \mid a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1 \}$

Intervals:

$a < b$

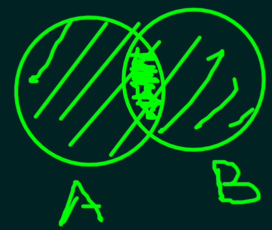
$[a,b] \to$ closed, $(a,b]$, $[a,b) \to$ semi-open $\to a \leq x < b$

$(a,b) \to$ open $= a < x < b$

Set operation:

A, B

$A \cup B = \{ x \mid x \in A \vee x \in B \}$



$A \cap B = \{ x \mid x \in A \wedge \underbrace{x \in B} \}$

$\bar{A} = \{ x \mid x \notin A \}$

$A - B = \{ x \mid x \in A \wedge (x \notin B) \}$

$= A \cap \bar{B}$

$A \triangle B$ = symmetric diff $= (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

$= (A \cap \bar{B}) \cup (B \cap \bar{A})$

De Morgans: $\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\qquad \bar{\bar{A}} = A$.

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

Associative: $A \cup (B \cup C) = (A \cup B) \cup C$

Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ✓

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Absorption: $A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$

$$x \in A \cup (B \cap C) = x \in A \vee \underbrace{x \in (B \cap C)}_{} \rightarrow \{x \in B \wedge x \in C\}$$

$$= \{x \in A \vee x \in B\} \wedge \{x \in A \vee x \in C\}$$

$$= x \in A \cup B \wedge x \in A \cup C$$

$$= x \in (A \cup B) \cap (A \cup C)$$

$x \in P \rightarrow x \in Q$

$P \subseteq Q$

$P \subseteq Q \quad Q \subseteq P$

$\Rightarrow P = Q$

+ Reverse direct.

▷ $A_1 \cup (A_1 \cap A_2) \cup (A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_3 \cap A_4) \cup \dots$

$= A_1$

▷ $\overline{A \triangle B} = \bar{A} \triangle B = A \triangle \bar{B}$?

$A - (B \cup C) = (A - B) \cap (A - C)$

Index Set: $A_1, A_2, \ldots \subseteq \mathcal{U}$,

I as index set, $i \in I$ s.t.

$\bigcup_{i \in I} A_i \leftarrow$ union $\qquad \bigcap_{i \in I} A_i \leftarrow$ intersect

Partition: $\longrightarrow$ (1) $\bigcup_{i \in I} A_i = \mathcal{U} \rightarrow$ Cover

(2) $A_i \underset{i \neq j}{\cap} A_j = \emptyset \longrightarrow$ disjoint

Ex: $Z_0 = \{3m \mid m \in \mathbb{Z}\} = \{0, \pm 3, \pm 6, \ldots\}$

$Z_1 = \{3m+1 \mid m \in \mathbb{Z}\} = \{\ldots, -8, -5, -2, +1, +4, +7 \ldots\}$

$Z_2 = \{3m+2 \mid m \in \mathbb{Z}\} = \{\ldots, -7, -4, -1, +2, +5, +8 \ldots\}$

$\mathbb{Z} \supseteq Z_i \qquad Z_1 \cup Z_2 \cup Z_3 = \mathbb{Z}$ and $Z_1 \cap Z_2 = \emptyset$

induces a partition over $\mathbb{Z}$. $\left[I = \{0, 1, 2\} \rightarrow \text{index Set} \right]$ $\qquad Z_2 \cap Z_3 = \emptyset$

$Z_3 \cap Z_1 = \emptyset$

Cartesian Product:

$A \times B = \{ (a, b) \mid a \in A,\ b \in B \}$

$(a, b) \neq (b, a)$ unless $a = b$

$B \times A = \{ (b, a) \mid b \in B,\ a \in A \}$

$A \times B \neq B \times A$

ordered pair

$A_1 \times A_2 \times \cdots \times A_k = \{ (x_1, x_2, \ldots, x_k) \mid x_i \in A_i \}$

$(n_1 n_2 \cdots n_k)$

$|A \times B| = mn = |B \times A|$

$n \quad m$

$= |A| * |B|$

$\hookrightarrow (a, b), (c, d) \in A \times B$

$(a, b) = (c, d)$ iff $a = c$ and $b = d$

Properties:

$A \times \phi = \phi \times A = \phi$

$A \times (B \cup C) = (A \times B) \cup (A \times C)$ —

$(A \cap B) \times C = (A \times C) \cap (B \times C)$ —

[(Binary) Relation]

$a \leftrightarrow b$

(a, b)

$A = \{1, 2\}$

$B = \{a, b, c\}$

$\rho_1 = \{ (1, a), (2, a), (1, b) \}$

$\rho_2 = \{ (1, a), (2, c) \} \ldots$

$\boxed{\rho \subset A \times B}$

$\# \text{Rel} = 2^{|A||B|}$

Properties: Reflexive: if $\forall x \in A$, $(x,x) \in \rho$. | $\rho \subseteq A \times A$

$(a,b) \in \rho \equiv a \rho b$

$A = \{1,2\}$

$\rho_1 = \{(1,1), (2,2)\}$ ✓

$(1,2)$ ✗

$\rho_2 = \{(1,1), (2,2)\}$ ✓ ✓

Symmetric: (ρ) if $\forall x, y \in A$,

$[x \rho y \longrightarrow y \rho x]$

$\rho = \{(1,1)\}$

$(x,y) \in \rho \longrightarrow (y,x) \in \rho$

Transitive: (ρ) if $\forall x,y,z \in A$

$(x,y) \in \rho$, $(y,z) \in \rho \Longrightarrow (x,z) \in \rho$

$A = \{1,2,3\}$

$\rho = \{(1,2), (2,3)\}$

$(1,3)$ ✓

Antisymmetric: (ρ) if

$\forall x,y \in A$, $(x,y), (y,x) \in \rho \longrightarrow x = y$

Irreflexive: $\exists x \in A$, $(x,x) \notin \rho$

Asymmetric: $\exists x, y \in A$, s.t. $[(x,y) \in \rho \wedge (y,x) \notin \rho]$

Non-trans: ______

Non-antisym: ______

Ex: (1) Ref + Sym, $\neg$ Trans: $\rho \subseteq \mathbb{Z} \times \mathbb{Z}$

$\rho = \{(x,y) \mid xy \geq 0,\ x,y \in \mathbb{Z}\}$

$x = 2$
$y = 0$ $z = -1$

(2) Sym + Trans, $\neg$ Ref: $\rho = \{(x,y) \mid xy > 0,\ x,y \in \mathbb{R}\}$

$$xz = \frac{(xy).(yz)}{y^2} > 0$$

$\longrightarrow x = 0$

(3) Refl + Trans, $\neg$ Sym: $\rho = \{(x,y) \mid x \leq y \text{ and } x,y \in \mathbb{R}\}$

$\longrightarrow x = 0.1 \quad y = 1.0$

(4) $\neg$ Ref, $\neg$ Sym, $\neg$ Trans: $\rho = \{(x,y) \mid y = x+1,\ x,y \in \mathbb{Z}\}$

$y = x+1 \not\Rightarrow x = y+1$

$\left.\begin{array}{l} y = x+1 \\ z = y+1 \end{array}\right\} \not\Rightarrow z = x+1$

Antisymmetric?? Yes

- Only Ref. $\rightarrow$
- Only Sym. $\rightarrow$
- Only Trans $\rightarrow$
- Only Anti Sy $\rightarrow$

H/W

▷ Equivalence Relation : $\rho \subseteq A \times A$

→ ① Reflexive
→ ② Symmetric
→ ③ Transitive

Ex: $\rho = \{ (x,y) \mid x-y \text{ is divisible by } 3,\ x,y \in \mathbb{Z} \}$ (over $\mathbb{Z}$)

↳ ① Reflex: $3 \mid (x-x)$ ✓

② Sym: $3 \mid (x-y) \rightarrow 3 \mid (y-x)$

as, $y - x = -(x-y)$ ✓

③ Trans: $3 \mid x-y$ & $3 \mid y-z$

$x - z = \underbrace{(x-y)} + \underbrace{(y-z)} \Rightarrow 3 \mid (x-z)$

↳ ρ is equivalence Relation

Trivial:

Ⓢ $(x,y) \in \rho \rightarrow (y,x) \in \rho$

$\boxed{T}$ $(x,y) \in \rho$, $(y,z) \in \rho$

↳ $(x,z) \in \rho$

$(x,x) \in \rho \Rightarrow R$

$\boxed{S \wedge T} \Rightarrow \boxed{R}$??

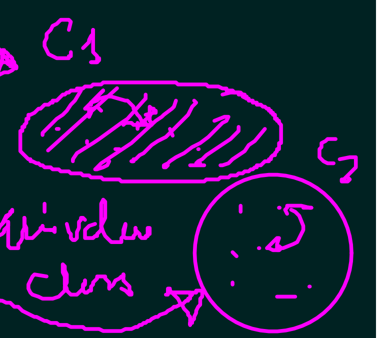
Fallacy!!

$\neg \exists y$ always which relates to x.

$Z_0 = \{ 3m \mid m \in \mathbb{Z} \} = \{ 0, \pm 3, \pm 6, \ldots \}$

$Z_1 = \{ 3m+1 \mid m \in \mathbb{Z} \} = \{ -5, -2, 1, +4 \ldots \}$

$Z_2 = \{ 3m+2 \mid m \in \mathbb{Z} \}$



$[3] = [6] = [-9] = [0]$

Equivalence class: $\rho \subseteq A \times A$ (equiv) $\Big\}$ A/ρ

$\forall y \in A \quad [y] = \{ x \mid (x,y) \in \rho,\ x \in A \}$

$\Rightarrow$ Property: ρ equiv rel over A then, ① $x \in [x]$

② $(x,y) \in \rho$ iff $[x] = [y]$

③ $[x] = [y]$ or $[x] \cap [y] = \phi$

Partition

Proof: ① $(x,x) \in \rho \quad x \in [x]$

② if $(x,y) \in \rho \quad a \in [x] \rightsquigarrow (a,x) \in \rho$ (T)

"$\Rightarrow$" $\Rightarrow (a,y) \in \rho \Rightarrow a \in [y]$

$b \in [y] \Rightarrow (b,y) \in \rho$

$(x,y) \in \rho$

$\Rightarrow (y,x) \in \rho$ (T)

$(b,x) \in \rho \quad \Rightarrow b \in [x]$

$\therefore [x] = [y]$

"$\Leftarrow$" if $[x] = [y] \quad x \in [x] = [y]$

$\Rightarrow x \in [y] \longrightarrow (x,y) \in \rho$

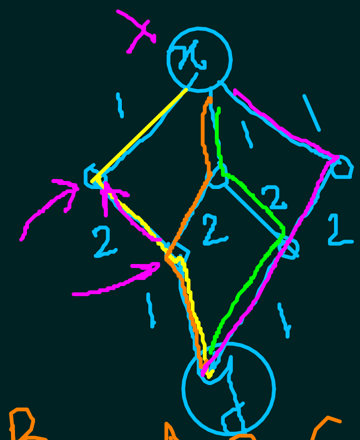
③ assume $[x] \neq [y]$, assume $[x] \cap [y] \neq \emptyset \quad u \in [x]$ and $u \in [y]$

$\perp$ Contradiction $\quad (x,y) \in \rho \longleftarrow (y,x) \in \rho \longleftarrow (u,x) \in \rho \quad (y,u) \in \rho$

▷ Partial Order Set (POSET):

$1 \leq 2 \leq 3 \leq \cdot\, n \cdots$ ⟶ total order

$n-1$



↳ $\rho \subseteq A \times A$ ρ as partial order relation

if Reflex, Antisym., Transitive

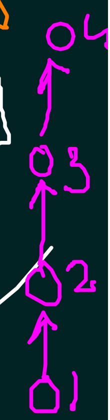
(A, ρ) a poset.

Ex. $S = \{1, 2, 3\}$ $\rho = \{(A, B) \mid A \subseteq B,\ A, B \in P(S)\}$ over $P(S)$

$(P(S), \subseteq)$ is Poset.

VS

$(\mathbb{Z}^+, \leq)$ poset?

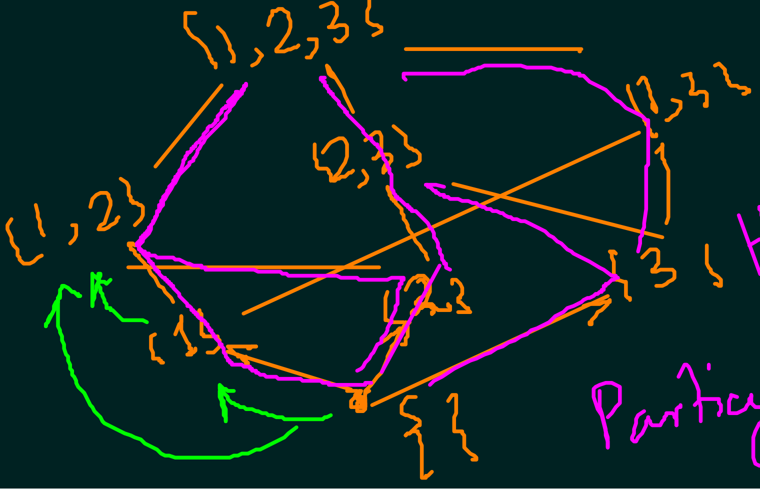


R $A \subseteq A$

AS $A \subseteq B$ and $B \subseteq A \rightarrow A = B$

T $A \subseteq B$ and $B \subseteq C \rightarrow A \subseteq C$

Covering relation



Hasse Diagram

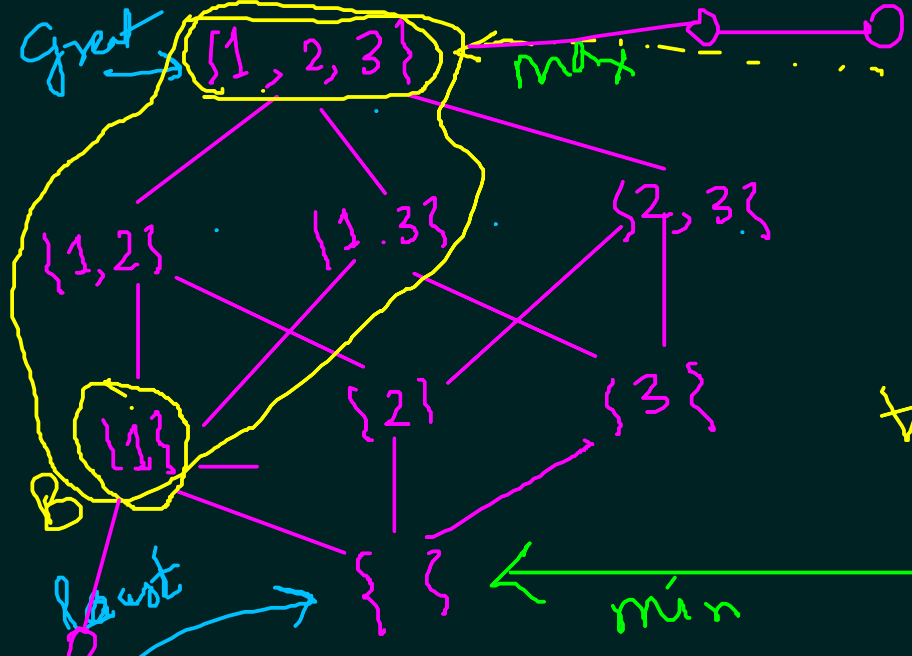
Partial Order

(A, ρ) poset $p, q, r \in A$ $(p \neq q \neq r)$

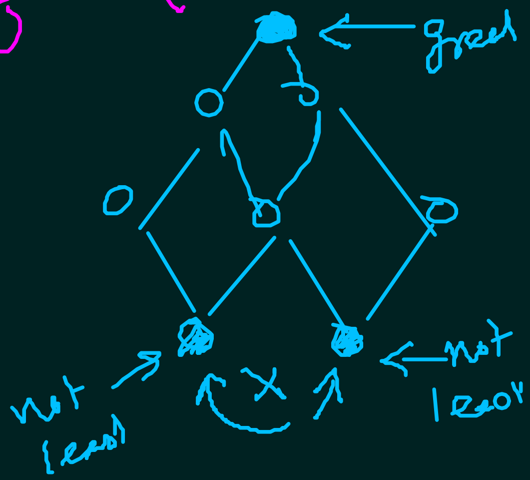
$p \lessdot q$ if $\neg \exists r$ s.t. $p < r < q$

$(p, q) \in \rho$ $(p, r) \in \rho$ $(r, q) \in \rho$

$\{\ \} \subseteq \{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\}$



$(P(S), \subseteq)$



Maximal element

(A, ρ) $\quad x \in A$ if

$\forall a \in A \; [a \neq x \rightarrow (x,a) \notin \rho]$

Minimal element

$y \in A$

if $\forall b \in A$

$[(b \neq y) \Rightarrow (b,y) \notin \rho]$

Least element

$x \in A$ in (A, ρ) poset

if $\forall a \in A \quad (x,a) \in \rho$

$\boxed{glb}$

Greatest element

$y \in A$ in (A, ρ)

if $\forall b \in A, \; (b,y) \in \rho$

$\boxed{lub}$

Lower Bound

(A, ρ) poset $\quad B \subseteq A$

$x \in A$ LB in B

if $\forall b \in B, \; (x,b) \in \rho$

Upper Bound

(A, ρ) poset $\quad B \subseteq A$

$y \in A$ is UB if

$\forall b \in B, \; (b,y) \in \rho$