SUMMARY

Predicate Logic : Formalism

- Constants : $\top, \bot, M, N, \ldots, 12, \ldots$
- Variables : $x, y, z, w, \ldots$
- Function Symbols : $F(x)$, $G(x,y,z)$
- Predicate Symbols : $p(x)$, $q(w,x,y)$
- Connectors : $\neg, \vee, \wedge, \rightarrow$
- Quantifiers : $\exists$ and $\forall$

Terms :

1. → Variables and Constants.
2. If $t_1, \ldots, t_k$ are terms and $F(x_1, \ldots, x_k)$ is Function Symbol then $F(t_1, \ldots, t_k)$ is Term.

Formula (WFF)

1. → Propositions
2. If $t_1, \ldots, t_k$ are Terms, and $p(x_1 \ldots x_k)$ is Predicate Symb. then $p(t_1, \ldots, t_k)$ is Predicate.
3. f_1 and f_2 are WFF, then $\neg f_1$, $f_1 \wedge f_2$, $f_1 \vee f_2$, $f_1 \rightarrow f_2$ are WFF.
4. If $p(x, \ldots)$ is Predicate (x free var) then $\forall x\, p(x, \ldots)$ and $\exists x\, p(x, \ldots)$ are WFFs.

Scopes :

→ Bound Variables
→ Free Variables

Ex: $\forall x [p(x, y) \wedge \exists y \exists z\, q(x, y, z, w)]$

bound free bound free

Deduction Mechanism:

To Prove, $(F_1 \wedge F_2 \wedge \ldots \wedge F_n) \rightarrow G$ is valid

We Prove, $(F_1 \wedge F_2 \wedge \ldots \wedge F_n \wedge \neg G)$ unsat.

Limitations:

→ Higher Order Logics :

$\forall p [(p(0) \wedge (\forall x (p(x) \rightarrow p(S(x))))) \rightarrow \forall y (p(y))]$

quantification over Pred/Funct

→ Temporal Logics:

next, future, always, until.

Expressibility : All Computable ~~or~~ Functions can be specified using Pred. Logic

Applications: Program Verification.

Resolution-Refutation in Propositional Logic:

$F_1 \wedge F_2 \wedge F_3 \rightarrow G$ is valid

$\equiv \underbrace{(F_1 \wedge F_2 \wedge F_3 \wedge \boxed{\neg G})}_{\vee,\ \wedge,\ \neg}$ unsat

Resolving: $\overset{C_i}{(\neg a \vee b)}$, $\overset{C_j}{(a \vee c)}$ $\rightarrow$ $(b \vee c)$

Refute: $(\neg d) \wedge d) \equiv \bot$

CNF $\rightarrow$ $\underbrace{(a \vee b \vee \neg c)}_{C_1} \wedge \underbrace{(c \vee d)}_{C_2} \wedge (\neg a \vee b)$

F

Rajesh either took the bus or came by cycle to class. If he came by cycle or walked to class he arrived late. Rajesh did not arrive late. Therefore he took the bus to class.

F_1: $(bus \wedge \neg cycle) \underline{\vee} (\neg bus \wedge cycle)$

F_2: $cycle \vee walk \rightarrow late$

F_3: $\neg late$ ← C_3

G: bus ? → $\neg G$: $\neg bus$ ← C_4

$F_1 \wedge F_2 \wedge F_3 \wedge \neg G$

$\boxed{C_{11}}$: $(bus \vee cycle)$

C_{12}: $(\neg cycle \vee \neg bus)$

F_2; $(\neg cycle \wedge \neg walk) \vee late$

C_{21}: $(\neg cycle \vee late)$

C_{22}; $(\neg walk \vee late)$

$\left\{ C_{11},\ C_{21} \rightarrow bus \vee late \right\}$

Res $\left\{ C_3,\ C_{21} \rightarrow \neg cycle \right\}$ ← c'

Res $\left\{ \neg cycle,\ C_{11} \rightarrow bus \right\}$ ← c''

$\left\{ bus,\ \neg G \rightarrow \bot \right\}$ Ref

(Sound & complete) → unsat, → sat

Resolution-Refutation in Predicate Logic:

$\rightarrow (F_1 \wedge \dots \wedge F_n \wedge \neg G)$

$$\forall x \left[\forall y \left(student(y) \rightarrow likes(x,y) \right) \rightarrow \exists z \left[likes(z,x) \right] \right]$$

$$\equiv \forall x \left[\neg \forall y \left(\neg stud(y) \vee likes(x,y) \right) \vee \exists z \left(likes(z,x) \right) \right] \quad \text{IFF}$$

$$\equiv \forall x \left[\exists y \left(stud(y) \wedge \neg likes(x,y) \right) \vee \exists z \left(likes(z,x) \right) \right] \quad \text{NNF}$$

$y = F(x)$ $\quad \forall x \quad z = G(x)$ $\quad$ Rename/Std.

$$\equiv \cancel{\forall} x \left[\left(stud(F(x)) \wedge \neg likes(x, F(x)) \right) \vee likes(G(x), x) \right]$$

Drop $\quad C_1 \quad C_2 \quad$ Skolemization

— Make into CNF —

Ex:

C_1: $\neg std(x,y) \vee pass(x,y)$ — $pass(Madan, z)$ $\cancel{x}$ ✓ Most General $\quad C'$

C_2: $stud(Madan, z)$ — — — $pass(Madan, Physics)$ ✗

C_3: $\neg pass(Rohan, Physics)$

C_4: $\neg pass(w, Math)$ ⟶ $\neg stud(w, Math)$ $\quad C''$

C_2 & $C'' \rightarrow \perp$ $\quad$ unification substitution

$F_1 : \forall x \, [\, cont(x) \rightarrow \neg dep(x) \,] \equiv \cancel{\forall x} \, [\, \neg cont(x) \vee \neg dep(x) \,]$ C_1

$F_2 : \exists x \, [\, eng(x) \wedge cont(x) \,] \longrightarrow eng(\overset{B}{A})_{C_2} \wedge cont(\overset{B}{A})_{C_3}$

$G : \exists x \, [\, eng(x) \wedge \neg dep(x) \,] \equiv$ $\neg G : \cancel{\forall x} \, [\, \neg eng(x) \vee dep(x) \,]$ C_4

Drop

$C_2 \quad C_4 \qquad dep(B)$

$dep(A) \qquad C_1$

$\neg cont(A) \qquad C_3$

?

$\perp$

$(F_1 \wedge \dots \wedge F_n \wedge \neg G)$

$\rightarrow$ unsat ✓

$\rightarrow$ sat

not "decidable"

(Sound)

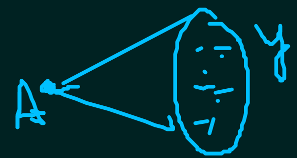
$x \quad \exists x \, q(x)$

$[\, q(A) \,]$ $A \in D_x$

Subst

$\forall x \, \exists y \, p(x, y)$

$\forall x \, p(x, F(x))$ ✓



$(e(A) \vee \perp)$

$(\neg e(A) \vee d(A))$

! Semi-Decidable !

$x \quad \exists x \, \forall y \, p(\overset{A}{\cancel{x}}, y) \longrightarrow \cancel{\forall y} \, p(A, y) \longrightarrow \underbrace{p(A, y)}$

- **Program Verif.**

$x > 0$

$y = 0$

```
if (x > 1)
    y ← y + 1
else
    y ← y - 1
```

y

$\forall x \, \exists y \, (\underbrace{x > 0}_{Pre}) \wedge \{P\} \rightarrow \{Post\}$

$(x > 1) \wedge (y' == y + 1)$ P_1

$(x \leq 1) \wedge (y' == y - 1)$ P_2

Post G

$\{Pre\} \wedge \{Path_1\} \rightarrow \{Post\}$ G ✓

$\{Pre\} \wedge \{Path_2\} \rightarrow \{Post\}$ ✓

$(x == qy + r) \wedge (r < y)$

- **Limitation:** Russel's Paradox / Barber's

① One barber

② $\exists x \, [per(x) \wedge \forall y \, [per(y) \rightarrow (sh(x, y) \Longleftrightarrow \neg sh(y, y))]]$

? Who shaves the barber?

expr.

Liar's Paradox: I am lying.