SUMMARY

▷ Propositional Logic:

- ↳ Constant (T, $\perp$)
- ↳ Boolean variables
- ↳ Connectives ($\wedge, \vee, \neg, \rightarrow$)

} Boolean Formula Encoding

1. Identify variables for simple statement / arguments.
2. Apply connectives to form Boolean Formula

$$\boxed{F_1 \wedge F_2 \wedge \cdots \wedge F_n \rightarrow G}$$

(valid) Tautology $\equiv$ All interpretations results TRUE

(falsify) Unsatisfiable $\equiv$ All interpret. results FALSE

▷ Satisfiable: At least one interpret. results TRUE

▷ Invalid: At least one interpret results FALSE.

▷ Deduction Procedure:

- ↳ Truth Table checking (exponential w.r.t. vars.)
- ↳ Inference Rules

$$\frac{p \qquad p \rightarrow q}{\therefore q}$$
Modus Ponens

$$\frac{\neg q \qquad p \rightarrow q}{\therefore \neg p}$$
Modus Tollens

$$\frac{p \rightarrow q \qquad q \rightarrow r}{\therefore p \rightarrow r}$$
Syllogism

$$\frac{p \vee q \qquad \neg p}{\therefore q}$$
Disjunctive Syllogism

$$\frac{p \wedge q \qquad p \rightarrow (q \rightarrow r)}{\therefore r}$$
Conditional Proof

$$\frac{p \rightarrow q \qquad r \rightarrow s \qquad p \vee r}{\therefore q \vee s}$$
Constructive Dilemma

$$\frac{p \rightarrow q \qquad r \rightarrow s \qquad \neg q \vee \neg s}{\therefore \neg p \vee \neg r}$$
Destructive Dilemma

ETC.....

Absorption Rules:
- $a \vee (a \wedge b) \equiv a$
- $a \wedge (a \vee b) \equiv a$

Rules are applied to reduce # vars

▷ Predicate Logic (First-Order):

- ↳ Boolean Variables extended to Boolean Predicates
- ↳ Constants extended to Functions.
- ↳ Quantifiers ($\forall$, $\exists$)

Today's Lecture

No contractors are dependable. Some engineers are contractors. Therefore, some engineers are not dependable.

F_1 $\forall x [\text{cont}(x) \to \neg \text{dep}(x)]$

$\exists x P(x) \equiv \neg \forall x [\neg P(x)]$

F_2 $\exists x [\text{engg}(x) \wedge \text{cont}(x)] \equiv \neg \forall x [\text{engg}(x) \to \neg \text{cont}(x)]$)))

$\neg \forall x [\text{cont}(x) \to \neg \text{engg}(x)]$

G $\exists x [\text{engg}(x) \wedge \neg \text{dep}(x)]$

✓ $T \equiv (F_1 \wedge F_2 \to G)$ or $(F_1 \wedge F_2 \wedge \neg G)$ is unsat ✓

Every passenger either travels in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore, some passengers travel in second class.

F_1 $\forall x [\text{pass}(x) \to (\cancel{\text{first}(x) \vee \text{sec}(x)})]$ ✓ exclusive

$(f(x) \wedge \neg s(x)) \vee (\neg f(x) \wedge s(x))$

F_2 $\forall x [\text{pass}(x) \to (s(x) \leftrightarrow \neg w(x))]$ ✓ ✓ $\equiv$

$\forall x [\text{pass}(x) \wedge s(x) \to \neg w(x)]$

$\forall x [\text{pass}(x) \wedge \neg w(x) \to s(x)]$

F_3 $\exists x [\text{pass}(x) \wedge w(x)]$

G: $\exists x [\text{pass}(x) \wedge s(x)]$

F_4 $\neg \forall x [\text{pass}(x) \to w(x)] \equiv \exists x [\text{pass}(x) \wedge \neg w(x)]$

F_4

Everyone likes everyone. $\forall x \; \forall y \; [\, like(x,y) \,]$ ✓

Someone likes someone. $\exists x \; \exists y \; [\, like(x,y) \,]$ —

Everyone likes someone. $\forall x \; \exists y \; [\, like(x,y) \,]$

Someone likes everyone. $\exists x \; \forall y \; [\, like(x,y) \,]$ ✓

Everyone is liked by everyone. $\boxed{\forall y \, \forall x} \; [\, like(x,y) \,]$ ✓

Someone is liked by someone. $\exists y \; \exists x \; [\, like(x,y) \,]$ —

Someone is liked by everyone. $\exists y \; \forall x \; [\, like(x,y) \,]$ ✓

Everyone is liked by someone. $\forall y \; \exists x \; [\, like(x,y) \,]$ ✓

$A \rightarrow S_1$

$B \rightarrow S_2$

not equivalent

$A \rightarrow S_i$

$B \rightarrow S_i$

$C \rightarrow S_i$

$like(x,y)$

▷ x likes y

Yes   No

If everyone likes everyone, then someone likes everyone. ← Tautology

$$\forall x \; \forall y \; (\, likes(x,y) \,) \rightarrow \exists x \; \forall y \; (\, likes(x,y) \,)$$

If some person is liked by everyone, then that person likes himself/herself. ← Tautology

$$\exists x \, [\, \forall y \; (\, likes(y,x) \,) \longrightarrow likes(x,x) \,]$$

If x is greater than y and y is greater than z, then x is greater than z.

$\forall x \forall y \forall z [gt(x, y) \wedge gt(y, z) \rightarrow gt(x, z)]$

$-ve$

The age of a person is more than the age of his/her child.

Func. symb. $Age(x)$ $gt(x, y)$ pred

$Age(\underline{y}) \wedge \underbrace{child(y, x)}$ T ⊥

$\forall x \forall y [child(y, x) \rightarrow gt(Age(x), Age(y))]$

The age of a person is greater than the age of his/her grandchild.

$a \rightarrow b$ T

FALSE

"vacuous"

$\forall x \forall y \forall z [\underbrace{child(y, x) \wedge child(z, y)}_{gc(z, x)} \rightarrow gt(Age(x), Age(z))]$

The sum of ages of two children is never more than or equal to the sum of ages of their parents.

— H/W —

$\forall x \; Pred(x, y)$

bound

free

$\forall x \, [P(x, y) \wedge \exists z \; q(x, y, z, w)]$

free

free

free

(Scope rule)

$\forall x \, [P(x, y) \wedge \exists y \, \exists z \; q(x, y, z, w)]$

$[\forall x \, [P(x)] \vee q(\cancel{y})$ ← free

$gt(Age(x), 12)$

$gt(a, b)$

Const : T, $\bot$, value, M, N,

($M = 12$, $N = 1.04$)

Var : x, y,

Pred : $pred(x, y)$, $q(y, z, w)$

Func : $Age(x)$, $F(x, y, z)$

Terms : Const, Var,

$F(t_1, \ldots, t_k)$ is a term.

Def. (Recursive)

$Sum(\underbrace{Age(x)}_{t_1}, \underbrace{Age(y)}_{t_2})$

Formula : prop. is Formula $\genfrac{}{}{0pt}{}{T}{\bot}$

$pred(t_1 \ldots t_k)$ Formula

$\forall x_1 \ldots \forall x_n \; pred(x_1 \ldots x_n)$

$\exists x_1 \ldots \exists x_n \; pred(x_1 \ldots x_n)$

formulas

$f_1 \wedge f_2$, $f_1 \vee f_2$, $f_1 \to f_2$, $\neg f_1$ formulas

Any Computable function (argument) can be expressed using P. L.

$X \to \boxed{P} \to Y \quad \forall x \exists y \{X\} \wedge P \to \{Y\} \leftarrow$ deduce

$\forall$ pred $\forall x$ (pred (x)) X

$\forall$ Age $\forall x$ (Age (x)) X

limitations → Higher Order logic

Example

$$\forall p \left[\left(p(0) \wedge \left(\forall x \left(p(x) \to p(S(x)) \right) \right) \right) \to \forall y \left[p(y) \right] \right]$$

— What is expressed ?

→ Russel's Paradox

→ Liar's Paradox

undecidable

→ Next class :

— deduction

— Limitations