TUTORIAL – 4

(UN-DECIDABILITY, REDUCIBILITY AND RICE'S THEOREM)

Problem-1

For a language L over the alphabet {0, 1}, define the language:

HALF(L) = { x | $x \in \Sigma^*$, and there exists $y \in \Sigma^*$

such that $|x| = |y|$ and $xy \in L$ }.

- Prove/Disprove the following statements.

(a) If L is R.E., then HALF(L) must be R.E.

(b) If L is Recursive, then HALF(L) must be Recursive.

Problem-2

Prove that the following language is not Recursive:

WB = { M # w | M writes the blank symbol in some step on input w }

Problem-3

Prove:

(a) E_{2026} = { M | M halts on exactly 2026 inputs }
is not R.E.

(b) AL_{2026} = { M | M halts on at least 2026 inputs }
is R.E. but not Rec.

Problem-4

Let k be a constant positive integer.

Consider the following languages:

- LE_k = { M | DTM M loops on at most k inputs },
- LT_k = { M | DTM M loops on less than k inputs },
- GE_k = { M | DTM M loops on at least k inputs },
- GT_k = { M | DTM M loops on more than k inputs }

Prove that all these languages are non-R.E. and non-co-R.E.

Problem-5

Let nsteps(M, w) denote the number of steps of M on w. If M loops on w, take nsteps(M, w) = ∞. If N also loops on v, take nsteps(M, w) = nsteps(N, v).

Prove: Recursive / R.E. but not recursive / non-R.E.?

(a) $L_a = \{ M \# N \mid \text{nsteps}(M, \varepsilon) < \text{nsteps}(N, \varepsilon) \}$

(b) $L_b = \{ M \# N \mid \text{nsteps}(M, \varepsilon) \leqslant \text{nsteps}(N, \varepsilon) \}$

(c) $L_c = \{ M \# N \mid \text{nsteps}(M, w) < \text{nsteps}(N, v) \text{ for some } w, v \}$

(d) $L_d = \{ M \# N \mid \text{nsteps}(M, w) < \text{nsteps}(N, v) \text{ for all } w, v \}$

Problem-6

Prove that the following languages are not Recursive.

(a) $L_1 = \{ M\#N \mid L(M) = L(N) \}$,

(b) $L_2 = \{ M\#N \mid L(M) \subseteq L(N) \}$,

(c) $L_3 = \{ M\#N \mid L(M) \cap L(N) = \emptyset \}$,

(d) $L_4 = \{ M\#N\#P \mid L(M) \cap L(N) = L(P) \}$

Problem-7

Prove that the following problems on a TM M are decidable.

(a) Decide whether M halts on some input within 2026 steps.

(b) Decide whether M on a given input w moves left at least 2026 times.

Problem-8

Use Rice's theorems to prove that neither the following languages nor their complements are R.E.

(a) FIN = { M | L(M) is finite }

(b) REG = { M | L(M) is regular }

Problem-9

Prove/Disprove:

No non-trivial property of R.E. languages is semidecidable.

THANK YOU !