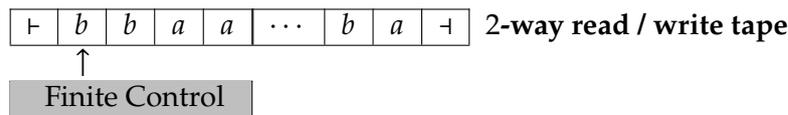


Tutorial 3

Computability Theory

Turing Machines, Equivalent Models and Unrestricted Grammars

1. Design a *total* Turing Machine (TM), \mathcal{K} , that accepts the language, $\mathcal{L}(\mathcal{K}) = \{0^{2^k} \mid k \in \mathbb{N}\}$.
2. A linear bounded automaton (LBA) is exactly like a 1-tape TM, except that the input string $x \in \Sigma^*$ is enclosed in left and right endmarkers \vdash and \dashv which may not be overwritten. The machine is constrained never to move left of \vdash or right of \dashv . It is allowed to read/write between these markers.



- (a) Give a rigorous formal definition of deterministic linearly bounded automata, including a definition of configurations and acceptance.
 - (b) Let M be an LBA with state set Q of size k and tape alphabet Γ of size m . How many possible configurations are there on input x with $|x| = n$?
 - (c) Argue that it is possible to detect in finite time whether an LBA loops on a given input. (*Hint*: Think of using Part (b).)
3. Show that the class of recursively enumerable sets is closed under union and intersection.
 4. Prove that, a language L is recursive if and only if there is an enumeration machine enumerating the strings of L in a non-decreasing order of length (string of the same length may be arranged in lexicographic order). For example, strings of $\{0, 1\}^*$ would be arranged as $0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \dots$
 5. Design an NTM to accept the language $\{wxyxz \mid w, x, y, z \in \{0, 1\}^*, |x| = 2026\}$.
 6. A TM \mathcal{M} has a two-way infinite tape. Initially, all cells on the tape are blank. Only one cell is storing the symbol $\#$. The head of \mathcal{M} is pointing to a blank. The task of \mathcal{M} is to locate the cell storing $\#$. Propose a strategy for doing this, (a) if \mathcal{M} is a DTM, (b) if \mathcal{M} is a NTM.
 7. Consider the following languages:
 - $L_a = \{a^n w c^n \mid w \in \{a, b, c\}^*, n \geq 0, \text{ and } \#a(w) = n\}$
 - $L_b = \{a^n w c^n \mid w \in \{a, b, c\}^*, n \geq 0, \text{ and } \#b(w) = n\}$
 - $L_c = \{a^n w c^n \mid w \in \{a, b, c\}^*, n \geq 0, \text{ and } \#c(w) = n\}$
 - $L_d = \{w c^m d^n \mid w \in \{a, b\}^*, m = \#a(w), \text{ and } n = \#b(w)\}$
 - $L_e = \{c^m w d^n \mid w \in \{a, b\}^*, m = \#a(w), \text{ and } n = \#b(w)\}$
 - $L_f = \{w \in \{a, b, c\}^* \mid \#a(w) = \#b(w) = \#c(w)\}$
 - $L_g = \{ww \mid w \in \{a, b\}^*\}$
 - $L_h = \{a^i b^j c^k d^l \mid i = k \text{ and } j = l\}$

where, $\#a(w)$, $\#b(w)$, and $\#c(w)$ denote the number of a 's, b 's, and c 's in the string w .

Answer the following.

- (a) Design separate TMs to accept each of the above languages ($L_a - L_h$).
- (b) Provide separate unrestricted grammars for each of the above languages ($L_a - L_h$).
- (c) Can you mechanically construct the unrestricted grammars (separately) for each of the above languages ($L_a - L_h$) from their corresponding TMs designed in Part (a)?
8. Consider the unrestricted grammar over the singleton alphabet $\Sigma = \{a\}$, having the start symbol S , and with the following productions.

$$S \rightarrow AS \mid aT, \quad Aa \rightarrow aaaA, \quad AT \rightarrow T, \quad T \rightarrow \epsilon$$

What is the language generated by this unrestricted grammar? Justify.

9. Prove that any grammar, defined over non-terminals N and terminals Σ , can be converted to an equivalent grammar with rules of the form $\alpha A \gamma \rightarrow \alpha \beta \gamma$ for $A \in N$ and $\alpha, \beta, \gamma \in (\Sigma \cup N)^*$.
10. A Jump Turing Machine (JTM) $\mathcal{J} = (Q, \Sigma, \Gamma, \delta, \vdash, \sqcup, s, t, r)$ is like a standard one-tape Turing Machine (TM) with the only exception that each transition of \mathcal{J} is of the form $\delta(p, A) = (q, B, m)$, where $p, q \in Q$, and $A, B \in \Gamma$, and $m \in \mathbb{Z}$. This means that if the finite control of \mathcal{J} is in the state p and the head of \mathcal{J} scans the tape symbol A , then the state changes to q , the content of the tape cell is changed from A to B , and the head jumps by m (integer) cells relative to the current position. If $m = 0$, the head stays at the current cell. If $m > 0$, the head makes a right jump. If $m < 0$, the head makes a left jump with the understanding that if the head is at position i on the tape and $|m| > i$, then the head goes to the leftmost cell (which stores the left end-marker \vdash). Also assume that if A is \vdash , then $m \geq 0$. Prove that a JTM is equivalent to a TM.
11. Let $f : \{1, 2, \dots, n\} \times \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n^2\}$ be the bijective function $f(i, j) = (i-1)n + j$ for $1 \leq i, j \leq n$. Consider the language L that consists of all strings $x \# 0^k$ of the following form.
- $x \in \{0, 1\}^*$, and $|x| = n^2$ for some integer $n \geq k \geq 1$. Denote $x = x_1 x_2 \dots x_{n^2}$.
 - For each i, j satisfying $1 \leq i < j \leq n$, we have $x_{f(i,j)} = x_{f(j,i)}$, and for each i in the range $1 \leq i \leq n$, we have $x_{f(i,i)} = 0$.
 - There exists a set $S \subseteq \{1, 2, \dots, n\}$ with $|S| \geq k$ such that for each $i, j \in S$, we have $x_{f(i,j)} = 0$.

Design a nondeterministic Turing machine (NTM) accepting L .

(Hint: Think of undirected graphs.)