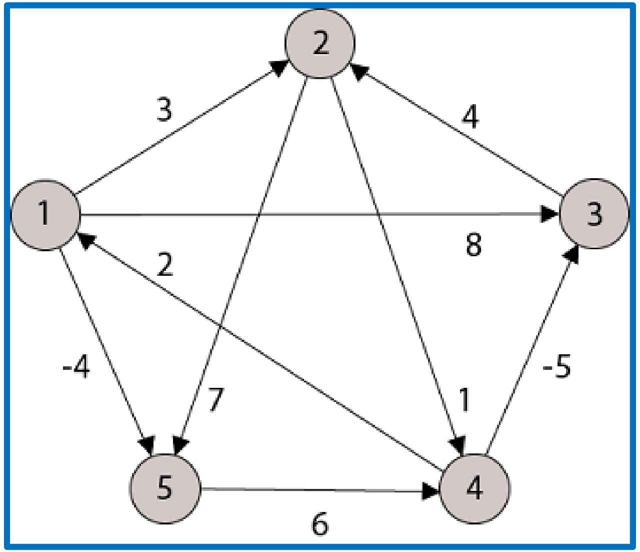
All-Pairs Shortest Path In A Graph



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Approaches to All-Pair Shortest Paths

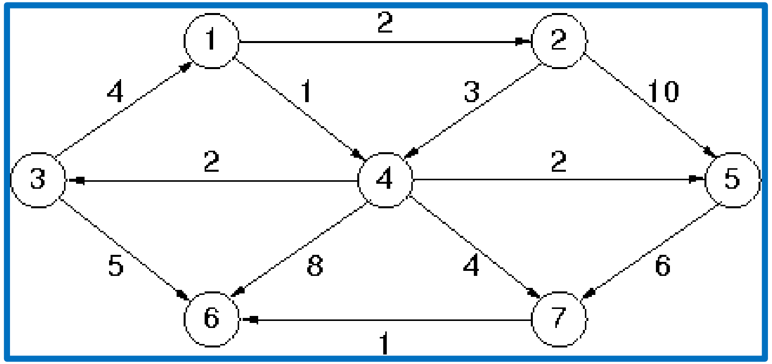
Problem: Given a weighted directed Graph G = (V, E), find the shortest (cost) path between all pairs of vertices in G.

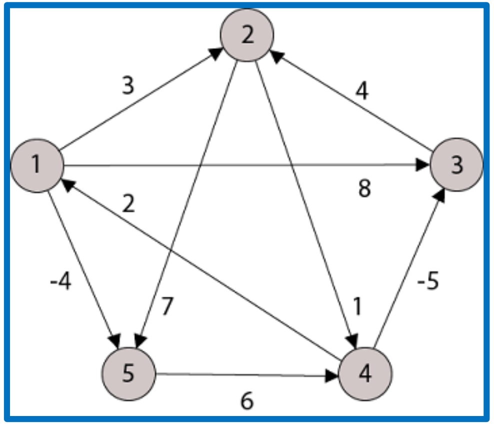
Case 1: For Directed Acyclic Graphs (DAGs), the recursive algorithm discussed earlier can be extended by computing the all-pair paths at every node during the recursion.

Case 2: For Graphs with positive edge costs, we can adapt the single source algorithm to continue to find the shortest path from s to all nodes (continue till OrQ is empty). We now repeat that for all nodes as source nodes.

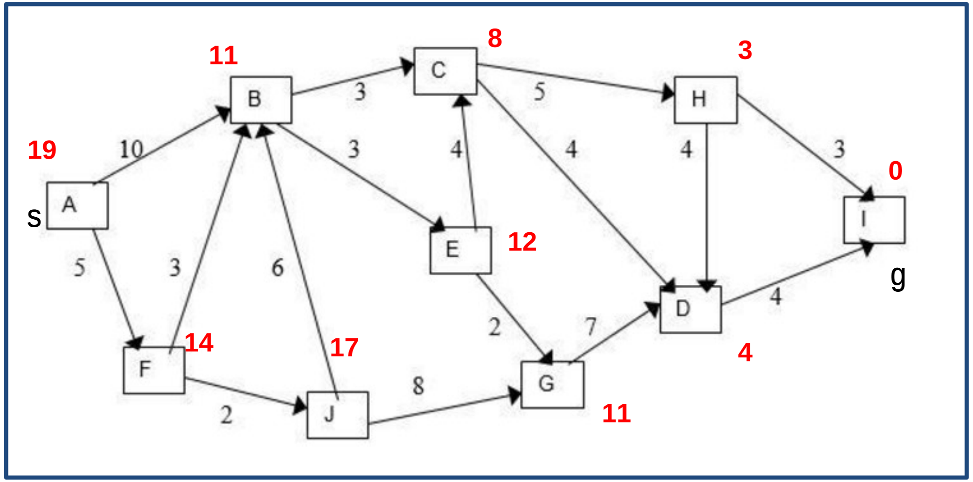
Case 3: For Graphs which may have negative edges but no negative edge cycles. We will discuss two methods, namely, Matrix Multiplication based method and the Floyd-Warshall Algorithm

Case 4: For graphs which may also have negative edge cycles, we will discuss the Bellman Ford Algorithm





Modifying Shortest Cost Path Algorithm for DAGs



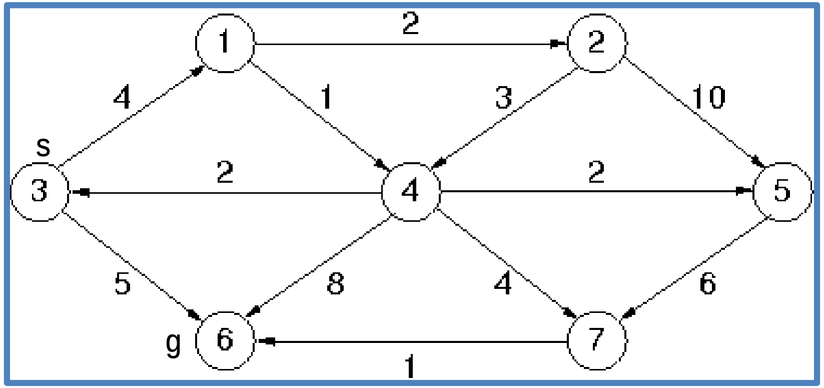
visited [i] indicates if node i is visited. / initially 0 /
cost[i] = cost of path from i to g, initially infinity
succ(i) = {set of nodes to which node i is connected}

```
DFSP(node,g) {
    local variable value = ∞;
    visited[node] = 1;
    if (node == g) { cost[node] = 0; return 0};
    for each n in succ(node) do {
        if (visited [n] == 0) DFSP(n);
        value = min (value, (cost[n] + C[node,n]))
        }
     cost[node] = value;
     return cost[node];
 }
```

Time Complexity O(|V| + |E|)
Will not work for Graphs which have cycles.
Works for negative edge cost DAGs.
Can be adapted to all pairs shortest paths for DAGs (Exercise).

Modifying the Best First Search Algorithm

```
G = (V,E) / Assume positive edge costs/
visited[i] all initialized to 0
cost[j] cost from s to j, all initialized to ∞
Ordered Queue OrQ initially {}
 BFSW(s,g) {
 cost [s] = 0; OrQ = {s};
 While OrQ != NULL {
     j = Remove_Min (OrQ); visited[j] = 1;
   if (j == g) terminate with solution cost[j];
   For each k in succ (j) {
   If (visited[k] == 0) {
          if (cost[k] > (cost[j] + C[j,k])) {
                 cost[k] = cost[j] + C[j,k];
                 Insert_Reorder(OrQ,k);}
          }
   } }
 If OrQ is empty terminate (“No Solution”);
} / This method is  called Dijkstra’s Algorithm /
```



	Queue OrQ with node costs	Node Removed
1	{1[0]}	1 [0]
2	{4[1], 2[2]}	4 [1]
3	{2[2], 3[3], 5[3], 7[5], 6[9]}	2 [2]
4	{3[3], 5[3], 7[5], 6[9]}	3 [3]
5	{5[3], 7[5], 6[8]}	5 [3]
6	{7[5],6[8])	7[5]
7	{6[6]}	6 [6]

Whenever a node is removed from OrQ, the best cost path to that node has been obtained. (Detailed proof is left as exercise)

Complexity is O(|E| log |E|), that is, O(|E| log |V|) using MinHeap or Balanced Tree. May also be implemented by an array in $O(|V|^2 + |E|)$

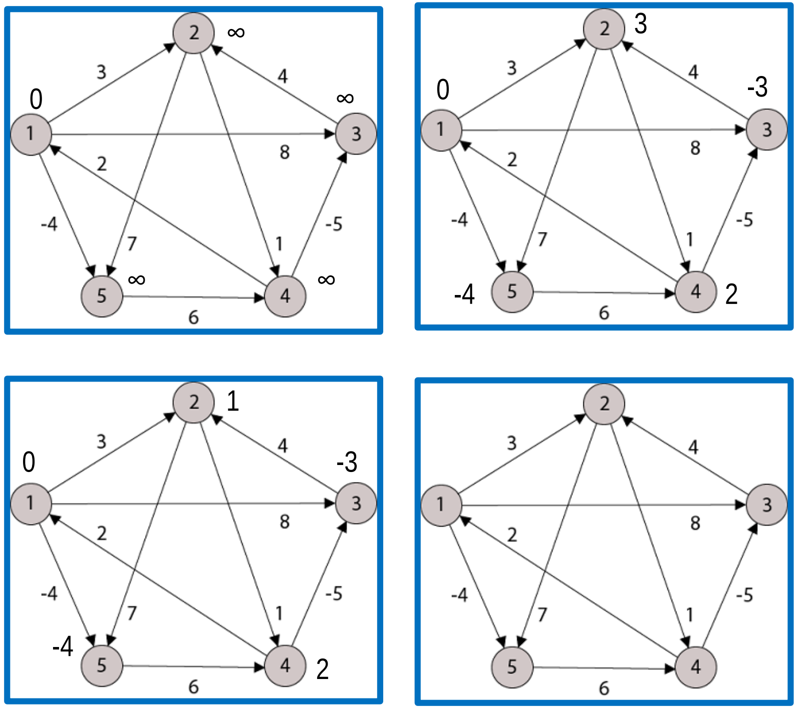
Bellman Ford Algorithm

visited [i] indicates if node i is visited. / initially 0 /
cost[i] = cost of path from i to g, initially infinity
succ(i) = {set of nodes to which node i is connected}
Parent[i] are parent pointers of shortest path, initialized to NULL

```
Bellman Ford(s) {
cost[s] = 0;
For i = 1 to |V| - 1 {
For each edge (n,k) in E {
        if (cost[k] > (cost[n] + C[n,k])) {
                cost[n] = cost[n] + C[n,k];
                 Parent[k] =n };
                  } }
For each edge (n,k) in E {
     if (cost[k] > cost[n] + C[n,k]) return ("Negative Cycle")
     }
 return("Success")
}
```

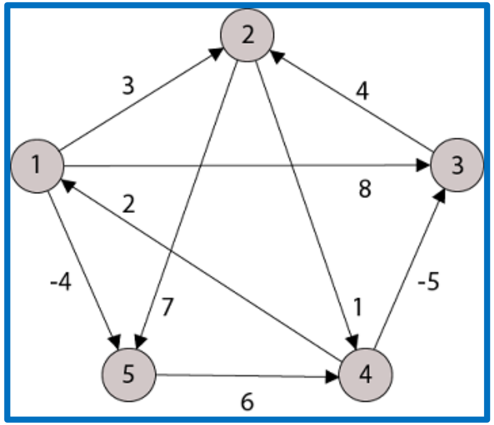
Time Complexity O(|E|*|V|) from s to all other nodes.
Works for negative edge cost graphs with negative edge loops.
For all-pairs, we run for each node as start node to get an O(|E| * $|V|^2$) Algorithm.

s = Node 1



Example taken from the book "Introduction to Algorithms" by Cormen, Leiserson, Rivest and Stein

Matrix Multiplication Based Method



RECURSIVE DEFINITION:

D[i,j,0] = 0 (if i = j), ∞ (if i != j)

D[i,j,k] = min { D[i,j,k-1], min { D[i,m,k-1] + C[m,j]}},

for all m in |V|

which is the same as: min { D[i,m,k-1] + C[m,j]}
since C[j,j] = 0 for all j;

Final Solution is D[i,j,n-1] where n = |V|

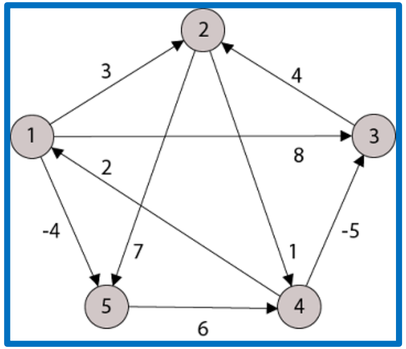
Analyzing the Recursive Definition we choose a Dynamic Programming Strategy using two 2-dimensional arrays D[n,n] for Memoization:

Top Down Recursive Scheme:

Bottom-up Iterative Scheme:

Time Complexity $O(|V|^4)$ time

Matrix Multiplication Based Method: Example



Example taken from the book "Introduction to Algorithms" by Cormen, Leiserson, Rivest and Stein

D[0]

0	∞	∞	∞	∞
∞	0	∞	∞	∞
∞	∞	0	∞	∞
∞	∞	∞	0	∞
∞	∞	∞	∞	0

D[1]

0	3	8	∞	-4
∞	0	∞	1	7
∞	4	0	∞	∞
2	∞	-5	0	∞
∞	∞	∞	6	0

D[2]

0	3	8	2	-4
3	0	-4	1	7
∞	4	0	5	11
2	-1	-5	0	-2
8	∞	1	6	0

D[3]

0	3	-3	2	-4
3	0	-4	1	-1
7	4	0	5	11
2	-1	-5	0	-2
8	5	1	6	0

D[4]

0	1	-3	2	-4
3	0	-4	1	-1
7	4	0	5	3
2	-1	-5	0	-2
8	5	1	6	0

RECURSIVE DEFINITION:

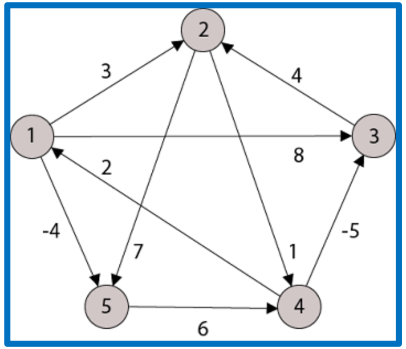
D[i,j,0] = 0 (if i = j), ∞ (if i != j)

D[i,j,k] = min { D[i,j,k-1], min { D[i,m,k-1] + C[m,j]}},
for all m in |V|

which is the same as: min { D[i,m,k-1] + C[m,j]}
since C[j,j] = 0;

Final Solution is D[i,j,n-1] where n = |V|

Improved Matrix Multiplication Based Method



RECURSIVE DEFINITION:

D[i,j,1] = 0 if (i==j)

= C[i,j] if (i!=j)

D[i,j,2k] = min { D[i,m,k] + D[m,j,k] },

for all m in |V|

Final Solution is D[i,j,n-1] where n = |V|

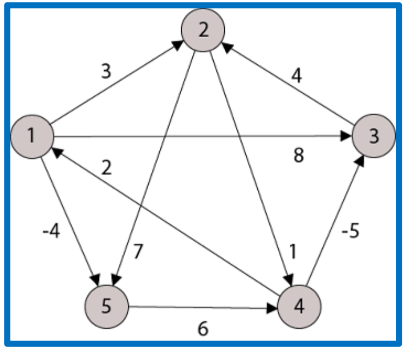
Analyzing the Recursive Definition we choose a Dynamic Programming Strategy using Two 2-dimensional arrays D[n,n] for Memoization:

Top Down Recursive Scheme:

Bottom-up Iterative Scheme:

Time Complexity $O(|V|^3 \log |V|)$ time

Improved Matrix Multiplication Based Method: Example



Example taken from the book “Introduction to Algorithms” by Cormen, Leiserson, Rivest and Stein

D[1]

0	3	8	∞	-4
∞	0	∞	1	7
∞	4	0	∞	∞
2	∞	-5	0	∞
∞	∞	∞	6	0

D[2]

0	3	8	2	-4
3	0	-4	1	7
∞	4	0	5	11
2	-1	-5	0	-2
8	∞	1	6	0

D[4]

0	1	-3	2	-4
3	0	-4	1	-1
7	4	0	5	3
2	-1	-5	0	-2
8	5	1	6	0

RECURSIVE DEFINITION:

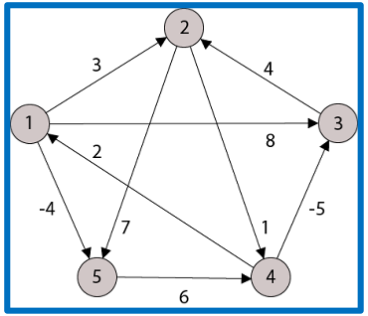
D[i,j,1] = 0 if (i==j)

= C[i,j] if (i!=j)

D[i,j,2k] = min { D[i,m,k] + D[m,j,k] }, for all m in |V|

Final Solution is D[i,j,n-1] where n = |V|

Floyd Warshall Algorithm



RECURSIVE DEFINITION:

F[i,j,0] = 0 if (i==j), and = C[i,j] otherwise

F[i,j,k] = min { F[i,j,k-1], F[i,k,k-1] + F[k,j,k-1]}

Final Solution is F[i,j,n] where n = |V|

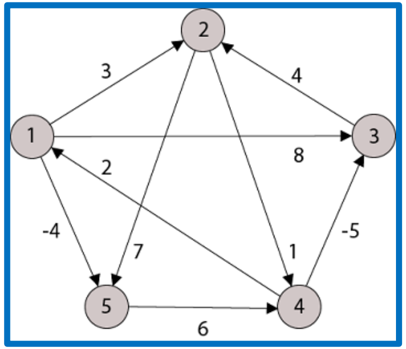
Analyzing the Recursive Definition we choose a Dynamic Programming Strategy using Two 2-dimensional arrays D[n,n] for Memoization:

Top Down Recursive Scheme:

Bottom-up Iterative Scheme:

Time Complexity O($|V|^3$) time

Floyd Warshall Algorithm: Example



Example taken from the book "Introduction to Algorithms" by Cormen, Leiserson, Rivest and Stein

F[0]

0	3	8	∞	-4
∞	0	∞	1	7
∞	4	0	∞	∞
2	∞	-5	0	∞
∞	∞	∞	6	0

F[1]

0	3	8	∞	-4
∞	0	∞	1	7
∞	4	0	∞	∞
2	5	-5	0	-2
∞	∞	∞	6	0

F[2]

0	3	8	4	-4
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2	5	-5	0	-2
∞	∞	∞	6	0

F[3]

0	3	8	4	-4
∞	0	∞	1	7
∞	4	0	5	11
2	-1	-5	0	-2
∞	∞	∞	6	0

F[4]

0	3	-1	4	-4
3	0	-4	1	-1
7	4	0	5	3
2	-1	-5	0	-2
8	5	1	6	0

F[5]

0	1	-3	2	-4
3	0	-4	1	-1
7	4	0	5	3
2	-1	-5	0	-2
8	5	1	6	0

RECURSIVE DEFINITION:

F[i,j,0] = 0 if (i==j) and = C[i,j] otherwise

F[i,j,k] = min { F[i,j,k-1], F[i,k,k-1] + F[k,j,k-1]}

Final Solution is F[i,j,n] where n = |V|

Summary: All-Pair Shortest Paths

Case 1: For Directed Acyclic Graphs (DAGs), the Recursive DFS Algorithm discussed earlier can easily be extended by computing the all-pair paths at every node. $O(|V|^2 + |V|*|E|)$

Case 2: For Graphs with positive edge costs, we can adapt the single source Best First Search (Dijkstra's) Algorithm to continue to find the shortest path from s to all nodes (Continue till OrQ is empty). We repeat that for all nodes as source nodes. $O(|V|*(|E| \log |V|))$

Case 3: For Graphs which may have negative edges but no negative edge cycles. We discussed two methods, namely,

Matrix Multiplication Based $O(|V|^3 \log |V|)$ and

Floyd-Warshall Algorithm $O(|V|^3)$

Case 4: For graphs which may also have negative edge cycles, we discussed the Bellman Ford Algorithm $O(|E| * |V|^2)$

Thank you