1. Formulate each of the below as a single statement (proposition or predicate), using only mathematical and logical notation that has been defined in class. For example, the use of logical quantifiers and connectives, and arithmetic, number-theoretic, and set-theoretic operations is allowed, as is the use of operators like gcd or sets like \( \mathbb{Q}, \mathbb{R} \), etc., but not the use of English-language words or informal shorthand like \{1,2,...,n\}.

(a) If \( a \) and \( b \) are integers and \( b \neq 0 \), then there is a unique pair of integers \( q \) and \( r \), such that \( a = qb + r \) and \( 0 \leq r < |b| \)

(b) Two integers are co-prime if and only if every integer can be expressed as their linear combination.

Answer:

(a) \( \forall a, b \in \mathbb{Z} : (b \neq 0) \rightarrow \left( \forall q, r, q', r' \in \mathbb{Z} : \left( (a = qb + r) \land (0 \leq r < |b|) \land (a = q'b + r') \land (0 \leq r < |b|) \right) \rightarrow (q = q') \land (r = r') \right) \)

(c) \( \forall a, n \in \mathbb{Z} : \left( \gcd(a, n) = 1 \right) \rightarrow \left( \forall b \in \mathbb{Z}, \forall x, x' \in \mathbb{Z} : \left( (ax \equiv_n b) \land (ax' \equiv_n b) \right) \rightarrow (x = x') \right) \)

2. For each of the following relations, state whether they fulfill each of the 4 main properties - reflexive, symmetric, anti-symmetric, transitive. Briefly substantiate each of your answers.

(a) The co-prime relation on \( \mathbb{Z} \). (Recall that \( a, b \in \mathbb{Z} \) are co-prime if and only if \( \gcd(a, b) = 1 \).)

(b) Divisibility on \( \mathbb{Z} \).

(c) The relation \( T \) on \( \mathbb{R} \) such that \( aTb \) if and only if \( a \cdot b \in \mathbb{Q} \).

Answer:

(a) It's definitely not reflexive, as no integer is co-prime with itself except -1 and 1. It is symmetric because \( \gcd(a, b) = \gcd(b, a) \), so \( \gcd(a, b) = 1 \) iff \( \gcd(b, a) = 1 \). Not anti-symmetric - every co-prime pair, such as (5,7) and (7,5), will show this. Not transitive - \( \gcd(5,7) = 1 \), \( \gcd(7,10) = 1 \), but \( \gcd(5,10) \neq 1 \).

(b) It's reflexive since any integer divides itself. Not symmetric, for example 2 divides 4 but 4 does not divide 2. It not anti-symmetric on \( \mathbb{Z} \), since \( a \mid -a \) and \( -a \mid a \), although it would be anti-symmetric if restricted to \( \mathbb{N} \). It is transitive | if \( a \mid b \) then \( b = ka \) for some \( k \in \mathbb{Z} \), and if \( b \mid c \) then \( c = lb \) for some \( l \in \mathbb{Z} \), thus \( c = (lk)a \) and \( (lk) \in \mathbb{Z} \) so \( a \mid c \).
3. In a partially ordered set (a non-strict partial order is a binary relation "≤" over a set \( P \) which is anti-symmetric, transitive, and reflexive), a chain is a totally ordered subset. For example, in the set \( \{1, 2, 3, 4, 5, 6\} \), the divisibility relation is a partial order and \( \{1, 2, 4\} \) and \( \{1, 3, 6\} \) are chains.

(a) What is the longest chain on the set \( \{1, 2, \ldots, n\} \) using the divisibility relation? How many distinct chains have this length? For the second part, make sure to consider all positive values of \( n \).

(b) What is the longest chain on the power set of a set \( \{\} \) with the \( \subseteq \) relation? How many distinct chains have this length?

Answer:

(a) Longest chain is powers of 2 as high as they can go, length is \( \log_2 n + 1 \). There is one chain of this length, except for \( n = 3 \) where there are two chains of length two.

(b) Each set in the chain must have distinct cardinality, so the longest chains are \( n + 1 \). The number of chains is the product of all binomial coefficients for \( n \) as they correspond to the number of sets of each cardinality.

4. Can a relation on a set be neither reflexive nor irreflexive [ \((a, a)\) does not belong to \( A \), if \( a \in A \)]? Give reasons.

Answer:

Digraph: [Diagram]

Neither Reflexive nor Irreflexive.

5. The relation \( R \) consisting of all pairs \((x; y)\) is such that \( x \) and \( y \) are bit strings of length three or more that agree in their first three bits. Is \( R \) an equivalence relation on the set of all bit strings of length three or more?

Answer:

1) Reflexivity: any string agrees with itself everywhere.

2) Symmetricity: If \( a \) agrees with \( b \) on any character after 4 then \( b \) also agrees with \( a \).
3) Transitivity: If a agrees with b and b agrees with c on some character then a agrees with c on this character, thus the property holds.

6. Give an example of a relation on a set that is (Use digraphs)
(a) symmetric and anti-symmetric
(b) neither symmetric nor anti-symmetric

**Answer:**

Draw your own digraphs.

a) Equality relation.

b) Suppose $aRb$ and $bRc$ and $cRb$. And that's as far as $R$ goes. It's not symmetric since (not $bRa$) and it's not anti-symmetric since both $aRb$ and $bRa$.

Or

The relation "divides" on the set $\mathbb{Z}$; The relation "preys on" in biological sciences.

7. Let $n$ be a positive integer and $S$ be a set of strings. $R_n$ is a relation on $S$ such that $sRnt$ for $s, t \in R_n$, if and only if $s = t$ or both $s$ and $t$ have at least $n$ characters and the first $n$ of them are the same. For instance, $01R301; 0011R300101$ hold but $01R3010; 01011R301110$ do not hold. Is $R_n$ an equivalence relation on $S$?

**Answer:**

We show that the relation $R_n$ is reflexive, symmetric, and transitive.

- **Reflexive:** The relation $R_n$ is reflexive because $s = s$, so that $sRns$ whenever $s$ is a string in $S$.
- **Symmetric:** If $sRnt$, then either $s = t$ or $s$ and $t$ are both at least $n$ characters long that begin with the same $n$ characters. This means that $tRns$. We conclude that $R_n$ is symmetric.
- **Transitive:** Now suppose that $sRnt$ and $tRnu$. Then either $s = t$ or $s$ and $t$ are at least $n$ characters long and $s$ and $t$ begin with the same $n$ characters, and either $t = u$ or $t$ and $u$ are at least $n$ characters long and $t$ and $u$ begin with the same $n$ characters. From this, we can deduce that either $s = u$ or both $s$ and $u$ are $n$ characters long and $s$ and $u$ begin with the same $n$ characters, i.e. $sRnu$. Consequently, $R_n$ is transitive.
- It follows that **$R_n$ is an equivalence relation.**