Trees and Distances

Tree: Characterization

- An n-vertex graph G (with $n \geq 1$) is a tree if and only if:
  - G is connected and has no cycles
  - G is connected and has $n - 1$ edges
  - G has $n - 1$ edges and no cycles
  - For all $u,v \in V(G)$, G has exactly one u,v-path
- Rooted Tree – a tree with one special vertex called root

Trees and Spanning Trees

- A graph having no cycles is *acyclic.*
- A *forest* is an acyclic graph.
- A *leaf* is a vertex of degree 1.
- A *spanning sub-graph* of G is a sub-graph with vertex set V(G).
- A *spanning tree* is a spanning sub-graph that is a tree.

Some Results …

- Every tree with at least two vertices has at least two leaves.
  - Deleting a leaf from a tree with *n* vertices produces a tree with *n*-1 vertices.

- If T is a tree with *k* edges and G is a simple graph with $\delta(G) \geq k$, then T is a sub-graph of G.

Some Results …

- If T and T′ are two spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.

Distances

- If G has a *u,v*-path, then the distance from *u* to *v*, written $d_G(u,v)$ or simply $d(u,v)$, is the least length of a *u,v*-path.
  - If G has no such path, then $d(u,v) = \propto$

Diameter and Radius

- The *eccentricity* of a vertex *u,* written $\varepsilon(u)$, is the maximum of its distances to other vertices.
- In a graph G, the *diameter,* diamG, and the *radius,* radG, are the maximum and minimum of the vertex eccentricities respectively.
- The *center* of G is the subgraph induced by the vertices of minimum eccentricity.
  - Informally, we often say just “vertices with minimum eccentricity”

Some Results

- The center of a tree is either a single vertex or an edge
- Equivalently
  - Every tree has either one or two centers.
  - If a tree has two centers, they must be adjacent

Median

- Let D(u) = sum of the distances from u to all other vertices in G
- Median = subgraph induced by vertices with minimum D value
  - As for centers, informally we often say “vertices with minimum sum of distances”
- The median of a tree is either a single vertex or an edge
  - Equivalently, a tree has one or two medians, and if it has two medians, they must be adjacent

Wiener Index

- Sum of the distances between all pairs of vertices in the graph
  - $= (1/2) \sum D(u)$
- For a tree T,

$$\text{Wiener Index} = \sum_{e=(u,v) \in E} n_u(e)n_v(e)$$

$n_u(e)$ = number of vertices in component containing u if edge e is removed from T

$n_v(e)$ = number of vertices in component containing v if edge e is removed from T

Counting Trees

- A *Labeled graph* is one in which each vertex is assigned an unique *label*
- How many labeled trees can be formed with n distinct vertices?
- Cayley's Theorem: There are n^{n-2} *possible* labeled trees with n distinct vertices

Prüfer Code / Sequence

Algorithm: *Production of* $f(T) = \{a_1, \ldots, a_{n-2}\}$

Input: A tree T with vertex set $S \subseteq \aleph$.

Iteration: At the i^{th} step, delete the least remaining leaf, and let a_i be the *neighbor* of this leaf.

Fundamental Cycles

- Branch – an edge in a given spanning tree
- Chord – an edge not in a given spanning tree
- Adding any chord to a spanning tree creates exactly one cycle, called *Fundamental Cycle*
- No. of fundamental cycles = no. of chords