Sketch of Solutions for Tutorial 7

Tutorial 7:

1. *Use the Konig-Egervary theorem to prove that every bipartite graph has a matching of size at least e(G)/Δ(G). Use this to prove that every subgraph of $K_{n,n}$ with more than (k-1)n edges has a matching of size at least k.* (5)

   By Konig-Egervary theorem, size of max. matching M = size of min vertex cover C. Now C $\geq$ e(G)/$\Delta(G)$ (as size of the min vertex cover is e(G)/)/$\Delta(G)$ even if all vertices are of degree Δ(G).). Therefore M $\geq$ e(g)/Δ(G).

   For any subgraph of $K_{n,n}$, $\Delta \leq n$. Since e > (k-1)n, therefore M > (k-1)n/n, or M > k-1. Therefore M $\geq$ k.

2. *Construct a graph that is neither a complete graph nor an odd cycle, but has a vertex ordering relative to which greedy coloring uses $\Delta(G)+1$ colors.* (5)

   Consider a path with 4 vertices, P = <a,b,c,d>. Take the vertices in the order a, d, b, c.

3. *Prove or disprove: For every graph G, chi$(G) \leq n(G) - \alpha(G) + 1$* (5)

   True. Consider the maximum independent set. Color it with one color. Then color all the remaining vertices with a distinct color. So G can be colored with $n(G) - \alpha(G) + 1$ colors. Hence the result.

4. *Prove or disprove: Every k-chromatic graph has some proper k-coloring in which some color class has $\alpha(G)$ colors.* (5)

   False. Consider the graph defined by

$$V = \{a, b, c, d, e, f\}$$

and

$$E = \{(a, c), (b, c), (c, d), (d, e), (d, f)\}$$

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Homework 7:

1. *A k-factor is a k-regular spanning subgraph of a graph G (so a perfect matching is a 1-factor because all vertices of the graph are in it (so it is a spanning subgraph) and degree of all vertices is 1 (so it is 1-regular)). Prove that every regular graph of positive even degree has a 2-factor. (Hint: Start with an Euler tour).* (10)

   Let G be any $2k$-regular connected graph ($k \geq 1$). The G contains an Euler tour $u_0, u_1, u_2, \ldots, u_l$, with $u_0 = u_l$. Now replace each vertex u with two vertices $u^+ and u^-$, and every edge $e = (u_i, u_{i+1})$ by the edge (u_i^+, u_{i+1}^-). the resulting bipartite graph (Take the + and the $-$ vertices as the two partite sets) G' is k-regular, and so has a 1-factor. Now collapse every vertex pair back into a single vertex to turn this 1-factor of G' into a 2-factor of G.

2. *Two people play a game on a graph G , alternately choosing distinct vertices. Player 1 starts the game by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.*

   Prove that the second player has a winning strategy if G has a perfect matching, and otherwise, the first player has a winning strategy. (A strategy is a particular choice of moves by a player. A strategy of a player is a winning strategy if it will always result in a win for the player, irrespective of what strategy the other player plays with). (5)

   Suppose G has a perfect matching. Then strategy for player 2: whenever first player choosea a node, player 2 chooses the node it is matched to. Think why this is a winning strategy for player 2 if there is a perfect matching in G.

   If G has no perfect matching, then strategy for player 1: Let M be a maximum matching in G. Player 1 first chooses a M-unsaturated vertex. Then onwards, whenever player 2 chooses a node, player 1 chooses its matching vertex in M. Think why this is a winning strategy for player 1 if there is no perfect matching in G.

3. *Teachers in a school have been asked to nominate, from those students known to them (every teacher may not know all students), three students each as candidates for the school council. At least one student nominated by each teacher is to be in Year 12. No student may be nominated by more than one teacher. The teachers decide to put their heads together to do the nomination. Can you help them in deciding if they can at all meet the nomination requirements, and if so, find such a nomination?* (5)

   Create three nodes for each teacher (for three roles of a teacher, one that will nominate a class 12 students and two that will nominate the other two), and a node for each student. Connect one copy of each teacher with all class 12 students only that they know, connect the remaining two copies with all students (including class 12). This is a bipartite graph with the teacher nodes on one side and the student nodes on the other. A solution exists if we can find a matching that saturates all teacher nodes, and the corresponding matching gives the nominations.

4. *Prove that $\chi(G) = \omega(G)$ if G' is bipartite.* (5)

   Let $G\prime = H$. Now, every independent set in a graph induces a clique in its complement, and vice versa. So $\omega(H') = \alpha(H')$. Also, $\chi(H')$ is the number of cliques in H needed to cover V(H). Since H is bipartite, these cliques must be edges. Hence, $\chi(H') = \beta'(H) = \alpha(H) = \omega(H')$ (using Konig's theorem). Now just note that $H' = G$.