Sketch of Solutions for Tutorial 6

Tutorial 6:

1. *Prove or disprove: Every tree has at most one perfect matching.*

   True.

   Note that existence of a perfect matching means the no. of nodes is even. Suppose a tree T has more than one perfect matching. Consider two such perfect matchings M_1 and M_2. Consider $M = M_1 \triangle M_2$. Then we know that the components of M can be even cycles or paths. However, in this case, no even cycles can be there as a tree has no cycles. Also, if a path of length > 0 if there, consider any one endpoint u of the path. Suppose that u is saturated by M_1 (i.e. the last edge of the path containing u is in M_1. Then, u is not saturated by M_2 as otherwise, an edge from M_2 would be incident on u in M. This is a contardiction, as M_2 is a perfect matching and hence saturates all vertices. Hence such a path also ccannot exists. Hence all paths in M are of 0 length, i.e., M consists of isolated vertices. Hence the two matchigs are the same.

2. *In a bipartite graph with partite sets X and Y, show that there exists a matching that saturates X if there exists a positive integer m such that degree of any vertex in X $\leq$ m $\leq$ degree of any vertex in Y.*

   Consider any subset S $\subseteq$ X of size r. The no. of edges incident on S is $\geq$ m.r. Each of these edges is incident on some edge in Y. Since the degree of any vertex in Y is $\leq$ m, the m.r edges are incident on at least (m.r)/m = r vertices in Y. Then by Hall's theorem, the result holds.

3. *Let G be a simple graph in which the sum of the degrees of any k vertices is less than (n-k). Prove that every maximal independent set in G has more than k vertices.*

Consider any independent set I of size $\leq$ k. Since the sum of the degrees of these k vertices is less than (n-k), there must be at least one vertex in G - I (which has n-k vertices) which does not have any edge from any of these k vertices. Hence it can be added to I to increase its size. Hence I is not maximal.